NEW FIRST COURSE IN THE THEORY OF EQUATIONS

BY

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In case a problem might offer some difficulty, it is preceded by a similar one solved in detail.

Answers are given to less than half the 850 problems. When no answer is given here, the problem does not occur (with answer) in *First Course.*

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NEW FIRST COURSE IN THE THEORY OF EQUATIONS

CHAPTER I

Complex Numbers

1. Square Roots. The positive square root of 3 is denoted by $\sqrt{3}$. In general, if p is any positive real number, the symbol $\sqrt{p}$ denotes the positive square root of p. It is easily computed by logarithms or by Horner's method (§ 61). If both p and q are positive, we therefore have

On the contrary, we shall express the square roots of negative real numbers in terms of the symbol i such that the relation $i^2=-1$ holds. Thus the roots of $x^2=-1$ are denoted by i and $-i$. The roots of $x^2=-9$ are written in the form $\pm 3i$ in preference to $\pm\sqrt{-9}$. If we insist unwisely on using the last notation we might be led to the erroneous conclusion that

where we have multiplied together the values -9 and -9 under the radical signs. The correct product is $3i \cdot 3i = 9i^2 = -9$.

In general, if p is real and positive, the roots of $x^2=-p$ are denoted by $\pm\sqrt{p}i$ and not by $\pm\sqrt{-p}$.

2. Complex Numbers. We shall call $3+4i$ a complex number and say that it is imaginary.

In general, if a and b are any two real numbers and $i^2=-1$, then $a+bi$ is called a *complex* number. Its *conjugate* is defined to be $a-bi$. If $b\neq 0$, $a+bi$ is said to be *imaginary*; in particular, bi is called a *pure imaginary* number. But if $b=0$, $a+bi$ becomes the real number a. Thus all real and all imaginary numbers are included among the complex numbers.

Two complex numbers $a+bi$ and $c+di$ are said to be *equal* if and only if $a=c$ and $b=d$. Since all real numbers (and hence zero) are special cases of $c+di$, this definition of equality implies that $a+bi=0$ if and only if $a=0$ and $b=0$.

Let a, b, c, and d be any real numbers. We define addition, multiplication, etc., of complex numbers as follows.

Addition: $(a+bi)+(c+di)=(a+c)+(b+d)i.$

Subtraction: $(a+bi)-(c+di)=(a-c)+(b-d)i.$

Multiplication: $(a+bi)(c+di)=(ac-bd)+(ad+bc)i.$

To find this product we first multiply the factors together in the usual manner and so obtain four terms, then replace i^2 by -1, and finally combine the terms.

Division: $$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}\,i,$$

where the last two fractions are real numbers. But $c^2+d^2=0$ would imply that $c=d=0$. Hence division by any complex number $c+di\neq 0$ is always possible and unique.

EXAMPLE 1. Express the square roots of $-7+24i$ as complex numbers.

Solution. We seek pairs of real numbers x and y for which

The square is x^2-y^2+2xyi. Hence $x^2-y^2=-7$, $2xy=24$. Squaring both and adding, we get $(x^2+y^2)^2=625$. Hence $x^2+y^2=25$. Combining this with $x^2-y^2=-7$, we get $x^2=9$, $y^2=16$. But $xy=12$ is positive. Hence $x=3$, $y=4$ or $x=-3$, $y=-4$. Therefore the square roots of $-7+24i$ are $\pm(3+4i)$.

PROBLEMS

Express as complex numbers

1. $\sqrt{-25}$.
2. $\sqrt{4}\sqrt{-16}$.
3. $(\sqrt{25}+\sqrt{-25})\sqrt{-16}$, *Ans.* $-20+20i$.
4. $8+2\sqrt{-5}$.
5. $8+2\sqrt{5}$.
6. $\dfrac{2}{i}$.
7. $\dfrac{3-\sqrt{-5}}{2-\sqrt{-1}}$.
8. $\dfrac{3+4i}{2-3i}$.
9. $\dfrac{a-bi}{a+bi}$.

10. Prove that the sum of two conjugate complex numbers is real and that their difference is a pure imaginary.

11. The conjugate of the sum of two complex numbers is equal to the sum of their conjugates. Does this result hold true if each word sum is replaced by the word difference?

12. The conjugate of the product of two complex numbers is equal to the product of their conjugates.

13. Solve Problem 12 when the word "product" is twice replaced by "quotient."

14. If the product of two complex numbers is zero, at least one of them is zero.

Express as complex numbers the square roots of

15. $11-60i$. **16.** $5+12i$. **17.** $-4gh+(2h^2-2g^2)i$. **18.** i, *Ans.* $\pm(1+i)/\sqrt{2}$.
19. $-i$. **20.** $24+70i$, *Ans.* $\pm(7+5i)$.

3. Geometrical Representation and Trigonometric Form of Complex Numbers. Using rectangular axes of coordinates OX and OY, we represent the complex number $a+bi$ by the point P having the (real) coordinates a and b (Fig. 1). The positive number $r=\sqrt{a^2+b^2}$ giving the length of OP is called the *absolute value* (or modulus) of $a+bi$. Let A denote the angle XOP, measured counter-clockwise from OX to OP. Then any of the angles A, $A\pm360°$, $A\pm720°$, $\cdots$ is called an *amplitude* of $a+bi$. Since $\cos A=a/r$, $\sin A=b/r$, we have

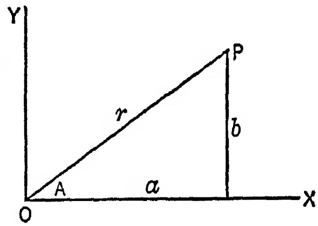


FIG. 1

$$a+bi=r(\cos A+i\sin A). \tag{1}$$

The second member is called the *trigonometric form* of $a+bi$.

Let $r'(\cos B+i\sin B)$ be a second complex number. Its product by (1) is rr' multiplied by

$$\begin{aligned}(\cos A+i\sin A)(\cos B+i\sin B)\\ =\cos A\cos B-\sin A\sin B+i(\sin A\cos B+\cos A\sin B)\\ =\cos(A+B)+i\sin(A+B).\end{aligned} \tag{2}$$

4. De Moivre's Theorem. *If* n *is any positive whole number,*

$$(\cos A+i\sin A)^n=\cos nA+i\sin nA. \tag{3}$$

Proof. This is trivial if $n=1$, and when $n=2$ it follows from (2) with $B=A$. To give a proof by mathematical induction, let (3) be true when $n=m$:

$$(\cos A+i\sin A)^m=\cos mA+i\sin mA.$$

Multiply each member by $\cos A+i\sin A$, and for the new second member substitute its value from (2) with $B=mA$. We get

$$(\cos A+i\sin A)^{m+1}=\cos(A+mA)+i\sin(A+mA),$$

which proves (3) when $n=m+1$. The induction is complete.

5. Cube Roots. To find the cube roots of a complex number, we first express it in its trigonometric form (1). The real cube root of the real number r may be found by logarithms (occasionally by inspection). Cube roots of the last factor in (1) may be found by using De Moivre's formula (3) with $n=3$ and A replaced by $\frac{1}{3}A$, $\frac{1}{3}(A+360°)$, and $\frac{1}{3}(A+720°)$, in turn.

$$(4)\qquad \begin{aligned} &(\cos\tfrac{1}{3}A+i\sin\tfrac{1}{3}A)^3=\cos A+i\sin A,\\ &\{\cos\tfrac{1}{3}(A+360°)+i\sin\tfrac{1}{3}(A+360°)\}^3=\cos A+i\sin A,\\ &\{\cos\tfrac{1}{3}(A+720°)+i\sin\tfrac{1}{3}(A+720°)\}^3=\cos A+i\sin A,\end{aligned}$$

since $\cos(A+360°)=\cos A$, $\cos(A+720°)=\cos A$, and similarly for sines.

Example 1. Find the cube roots of

$$(5)\qquad 4\sqrt{2}+4\sqrt{2}i=8(\cos 45°+i\sin 45°).$$

Solution. Since $r=8$, whose real cube root is 2, the answers are the doubles of the cube roots of $\cos 45°+i\sin 45°$. By (4), with $A=45°$, the latter has the cube roots

$$(6)\qquad \cos 15°+i\sin 15°,\quad \cos 135°+i\sin 135°,\quad \cos 255°+i\sin 255°.$$

The numbers (6) are distinct since they have the respective amplitudes 15°, 135°, 255°. But an equation $x^3=c$ has at most three distinct roots (§ 13). Hence the doubles of the numbers (6) give all the cube roots of the number (5).

[illegible] 2. Find the cube [illegible]

Solution. By (4) with $A=0$, the cube roots of unity are

$$(7)\qquad [illegible]^2=\cos 240°+i\sin 240°=-\tfrac{1}{2}-[illegible]$$

PROBLEMS

1. By factoring x^3-1 show that the cube roots of unity are 1 and the roots of $x^2+x+1=0$. Solve this equation and hence check (7).

2. Find by the method of Ex. 2 the fourth roots of unity. Find them also by solving $x^4-1=0$ algebraically.

3. Find the three cube roots of -64, and those of $-8i$.

4. Find the cube roots of ω. *Ans.* $R=\cos 40°+i \sin 40°$, ωR, $\omega^2 R$.

5. Find the cube roots of $4\sqrt{3}+4i$, and those of $4+4\sqrt{-3}$.

6. Without computation, find the square roots of ω.

7. The absolute value of the product of two complex numbers is equal to the product of their absolute values, while an amplitude of the product is equal to the sum of their amplitudes.

8. If $a+bi$ and $c+di$ are represented by the points A and C in Fig. 2, prove that their sum is represented by the fourth vertex S of the parallelogram two of whose sides are OA and OC. Hence show that the modulus of the sum of two complex numbers is equal to or less than the sum of their moduli, and is equal to or greater than the difference of their moduli.

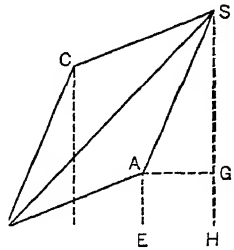


Fig. 2

9. If $a+bi$ and $e+fi$ are represented by the points A and S in Fig. 2, prove that the complex number obtained by subtracting $a+bi$ from $e+fi$ is represented by the point C. Hence show that the absolute value of the difference of two complex numbers is equal to or less than the sum of their absolute values, and is equal to or greater than the difference of their absolute values.

CHAPTER II

Elementary Topics

6. Quadratic Equation. If a, b, c are given (complex) numbers,

(1) $$ax^2+bx+c=0 \qquad (a\neq 0)$$

is called a *quadratic equation* or equation of the second degree. The reader is familiar with the method of "completing the square" to find its roots r and s:

(2) $$r=\frac{-b+\sqrt{D}}{2a}, \quad s=\frac{-b-\sqrt{D}}{2a}, \quad D=b^2-4ac.$$

We call D the *discriminant* of equation (1) and also the discriminant of the function ax^2+bx+c. In formulas (2), we employ $\sqrt{D}$ to be a definitely chosen complex number whose square is D(§§ 1, 2). We find at once that

(3) $$r+s=-\frac{b}{a}, \qquad rs=\frac{c}{a}.$$

Hence for all values of the variable v,

(4) $$a(v-r)(v-s)=av^2-a(r+s)v+ars\equiv av^2+bv+c,$$

the sign $\equiv$ being used instead of $=$, since these functions of v are *identically equal*, the coefficients of like powers of v being the same.

We speak of $a(v-r)(v-s)$ as the *factored form* of the function av^2+bv+c, and call $v-r$ and $v-s$ its *linear factors.*

In formula (4) we assign to v the values r and s in turn and get

$$0=ar^2+br+c, \qquad 0=as^2+bs+c,$$

so that the numbers (2) are actually the roots of equation (1).

7. Geometrical Solution of a Quadratic Equation. If in a proposed equation (1), we divide all terms by a and obtain an equation of the form $x^2-gx+h=0$. Let g and h be real. Fig. 3 shows the points $B=(0, 1)$ and $Q=(g, h)$. Draw the circle having BQ as a diameter. Its center is $(\frac{1}{2}g, \frac{1}{2}(h+1))$. The square of BQ is $g^2+(h-1)^2$. Hence the equation of the circle is

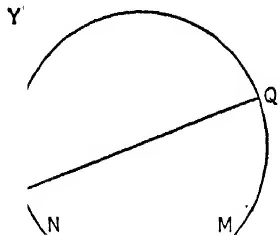


FIG. 3

2 /

When $y=0$, this reduces to $x^2-gx+h=0$. Hence if the x-axis intersects the circle in two distinct points N and M, their abscissas ON and OM are the two distinct real roots of $x^2-gx+h=0$. If the circle is tangent to the x-axis, so that the points N and M coincide, the roots are real and equal. But if the circle does not intersect the x-axis, the roots are imaginary. The latter is evidently true also when Q coincides with B, so that $l=0$, a case tacitly excluded from the above discussion.

PROBLEMS

Discuss geometrically:

1. $x^2-7x+12=0$. **2.** $-4=0$, *Ans.* 0.7, -5.7, approximately.

3. $x^2-5x-4=0$. **4.** $x^2-6x+9=0$.

5. $x^2-6x+16=0$.

6. To find $\sqrt{p}$, when $p>0$, take $g=0$, $h=-p$. In Fig. 3, Q is now on the prolongation below O of line BO, while N is at the left of O. Hence if BO and OQ are juxtaposed segments of a line and are of lengths 1 and p, respectively, the perpendicular to BOQ at O intersects the circle having BQ as diameter in two points N and M such that $OM=\sqrt{p}$, $ON=-\sqrt{p}$. See Fig. 4.

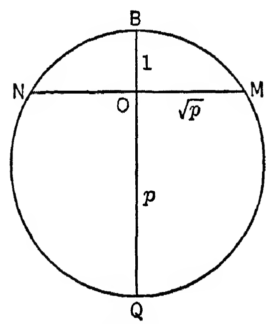


FIG. 4

7. What theorem in geometry proves at one step the result in Problem 6?

8. Construct $\sqrt{5}$. **9.** Construct $\sqrt{7}$.

8. Polynomial. Expressions like $2x^2+3$, x^3-x+5, and

are called *polynomials* in x, provided $a, \cdots, l$ are all constant complex numbers or in any case are quantities independent of x. We shall often denote the polynomial (5) by an abbreviated notation like $f(x)$. It is said to be of *degree* n if $a \neq 0$. It is called a *real polynomial* if its coefficients $a, b, \cdots$ are all real.

If $a \neq 0$, $f(x)=0$ is an equation of degree n, when $f(x)$ stands for the polynomial (5). If $n=2$, it is usually called a *quadratic equation* (§ 6); if $n=3$, a *cubic equation*; if $n=4$, a *quartic equation*.

9. The Remainder Theorem. *When* c *is a constant and a polynomial* f(x) *is divided by* x−c *until a remainder independent of* x *is obtained, this remainder is equal to* f(c), *which is the value of* f(x) *for* x=c.

For example, let $f(x)$ be x^3+3x^2-2x-5 and let $c=2$. To divide $f(x)$ by $x-2$ use the process called "long division."

$$
\begin{array}{r|r|l}
x-2 & x^3+3x^2-\ 2x-\ 5 & x^2+5x+8=\text{quotient } q(x) \\
 & x^3-2x^2 \qquad\qquad & \\
\hline
 & 5x^2-\ 2x-\ 5 & \\
 & 5x^2-10x \quad & \\
\hline
 & 8x-\ 5 & \\
 & 8x-16 & \\
\hline
 & 11=\text{remainder } r. &
\end{array}
$$

Instead of subtracting from $f(x)$ the multiples $x^3-2x^2=x^2(x-2)$, $5x^2-10x=5x(x-2)$, $8x-16=8(x-2)$ in succession, we evidently obtain the same result, 11, if we subtract $(x^2+5x+8)(x-2)$ or $(x-2)\,q(x)$. Hence

To prove the theorem, denote the remainder by r and the quotient by $q(x)$. Since the dividend is $f(x)$ and the divisor is $x-c$, the familiar "long division" process in algebra consists in brief in subtracting the product of $x-c$ by $q(x)$ from $f(x)$ and the difference is the remainder r. Transposing the product, we get

$$f(x) \equiv (x-c)q(x)+r, \tag{6}$$

identically in x. Hence we may take $x=c$ in (6) and get $f(c)=r$.

In case $r=0$, the division of $f(x)$ by $x-c$ is exact. Hence we have proved also the following useful result:

THE FACTOR THEOREM. *If* f(c) *is zero, the polynomial* f(x) *has the factor* x − c. *In other words, if* c *is a root of* f(x) = 0, *then* x − c *is a factor of the polynomial* f(x).

For example, 2 is a root of $x^3-8=0$, so that $x-2$ is a factor of x^3-8. Another illustration is furnished by formula (4).

PROBLEMS

1. If the discriminant $D=b^2-4ac$ of equation (1) is zero, the roots (2) are equal, so that, by formula (4), av^2+bv+c is the square of $\sqrt{a}(v-r)$. Prove conversely that the latter implies $D=0$.

2. Let equation (1) be real (a, b, c all real numbers). If D is positive, the roots (2) are both real. But if D is negative, the roots are conjugate imaginaries.

3. Illustrate Problems 1 and 2 for $x^2-2x+c=0$ when $c=1$, $c=0$, $c=2$, in turn.

4. Verify that $x^2-x+1+i=0$ has the root i. Find the second root by use of (3). Are the roots conjugate imaginaries?

5. Construct a quadratic equation the sum of whose roots is 3 and the product is 5. Is there a single answer?

6. Find the factored form of x^2-5x+6.

Without actual division find the remainder when

7. x^2-5x+6 is divided by $x-4$.

8. x^3-3x^2+6x-5 is divided by $x-3$. *Ans.* 13.

9. x^4-3x^2-2x-4 is divided by $x+3$.

Without actual division show that

10. x^2-5x+6 is divisible by $x-2$.

11. $13x^{10}+14x^5+1$ is divisible by $x+1$.

12. $2x^4-x^3-6x^2+4x-8$ is divisible by both $x-2$ and $x+2$.

13. $v^4-3v^3+3v^2-3v+2$ is divisible by both $v-1$ and $v-2$.

14. r^4-1, r^5-1, r^6-1 are divisible by $r-1$.

15. Verify by multiplication that

16. Hence prove that the sum of the numbers a, ar, ar^2, $\cdots$, ar^{n-1} in geometrical progression (with $r\neq 1$) is

$$r-1$$

17. A positive whole number p (like 5 and 7) is called a *prime* if p and 1 are the only positive whole numbers which divide p. Prove that the sum of the divisors of p^{n-1} is

$$\frac{p^n-1}{p-1}.$$

18. At the end of each of n years a man deposits a dollars in a savings bank. With annual compound interest at 4%, show that his account at the end of n years will be

$$\frac{a}{.04}\{(1.04)^n-1\}\text{dollars.}$$

19. In Problem 15 take $r=x/y$, clear of fractions, and derive

20. In Problem 19 change the sign of y. Write down the resulting identity when n is odd and when n is even. Check by the factor theorem.

21. Hence find (without division) the quotient of x^5+y^5 by $x+y$.

10. Synthetic Division. The labor of computing the value of a polynomial in x for an assigned value of x may be shortened by a simple device. To find the value of

for $x=2$, note that $x^4=x\cdot x^3=2x^3$, so that the sum of the first two terms of the polynomial is $5x^3$. To $5x^3=5\cdot 2^2x$ we add the next term $-2x$ and obtain $18x$ or 36. Combining 36 with the final term -5, we obtain the desired value 31.

This computation may be arranged systematically as follows. After supplying zero coefficients of missing powers of x, we write the coefficients in a line, ignoring the powers of x.

$$\begin{array}{ccccc} & 3 & 0 & -2 & -5 \\ & 2 & 10 & 20 & 36 \\ 1 & 5 & 10 & 18 & 31 \end{array}$$

First we bring down the first coefficient 1. Then we multiply it by the given value 2 and enter the product 2 directly under the second coefficient 3, add and write the sum 5 below. Similarly, we enter the product of 5 by 2 under the third coefficient 0, add and write the sum 10 below; etc. The final number 31 in the third line is the value of the polynomial when $x=2$. The remaining numbers in this third line are the coefficients, in their proper order, of the quotient

which would be obtained by the ordinary long division of the given polynomial by $x-2$.

We shall now prove that this process, called *synthetic division*, enables us to find the quotient and remainder when any polynomial $f(x)$ is divided by $x-c$. Write

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n$$

and let the constant remainder be r and the quotient be

$$q(x) \equiv b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1}.$$

By comparing the coefficients of $f(x)$ with those in

$$(x-c)\ q(x) + r \equiv b_0x^n + (b_1 - cb_0)x^{n-1} + (b_2 - cb_1)x^{n-2} + \cdots + (b_{n-1} - cb_{n-2})x + r - cb_{n-1},$$

we obtain relations which become, after transposition of terms,

$$b_0 = a_0, \quad b_1 = a_1 + cb_0, \quad b_2 = a_2 + cb_1, \quad \ldots, \quad b_{n-1} = a_{n-1} + cb_{n-2}, \quad r = a_n + cb_{n-1}.$$

The steps in the work of computing the b's may be tabulated as follows:

$$\begin{array}{cccccc|l} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & \underline{c} \\ & cb_0 & cb_1 & \cdots & cb_{n-2} & cb_{n-1} & \\ \hline b_0 & b_1 & b_2 & \cdots & b_{n-1}, & r & \end{array}$$

In the second space below a_0 we write b_0 (which is equal to a_0). We multiply b_0 by c and enter the product directly under a_1, add and write the sum b_1 below it. Next we multiply b_1 by c and enter the product directly under a_2, add and write the sum b_2 below it: etc.

PROBLEMS

In Problems 1, 2, 3, find r and q by synthetic division when we

1. Divide x^3+3x^2-2x-5 by $x-2$. *Ans.* $r=11$, q
2. Divide $2x^5-3x^3+2x+1$ by $x+2$.
3. Divide $x^3+6x^2+10x-1$ by $x-0.09$. *Ans.* $r=-.05067$, $q=x^2+6.09x+10.5481$.
4. Find the quotient of $x^3+4x^2-47x-210$ by x^2-x-42.
5. Find the quotient of $x^4-x^3-12x^2+16x-64$ by x^2-16.
6. Find the quotient of $x^4-3x^3+3x^2-3x+2$ by x^2-3x+2. *Ans.*
7. Solve Problems 7–9, 12, and 13 in § 9 by synthetic division.

11. Depressed Equation. Consider

(7)

If $f(x)=0$ has the root r_1, the factor theorem shows that $f(x)$ has the factor $x-r_1$, so that

(8) $$f(x)\equiv(x-r_1)Q(x), \quad Q(x)=ax^{n-1}+Dx^{n-2}+\cdots+K.$$

The coefficients of $Q(x)$ are rapidly computed by synthetic division. Every root of $Q(x)=0$ is evidently a root of $f(x)=0$. Conversely, if r_2 is a root, distinct from r_1, of $f(x)=0$, then r_2 is a root of $Q(x)=0$. In fact, the identity (8) holds when $x=r_2$ and then gives $0=(r_2-r_1)\,Q(r_2)$. Since $r_2-r_1\neq 0$, this implies $Q(r_2)=0$.

When one root r_1 of $f(x)=0$ is known, it is usually more difficult to find its further roots $r_2, r_3, \cdots$ directly than to obtain them as the roots of the *depressed equation*

(9) $$Q(x)\equiv\frac{f(x)}{x-r_1}=0$$

of degree $n-1$.

If $Q(x)=0$ has the root r_2, then, as in (8),

$$Q(x)\equiv(x-r_2)\,R(x), \qquad R(x)=ax^{n-2}+Mx^{n-3}+\cdots+T.$$

Inserting this expression for $Q(x)$ into identity (8), we get

(10) $$f(x)\equiv(x-r_1)(x-r_2)R(x).$$

Every root of $R(x)=0$ is evidently a root of $f(x)=0$. Conversely, if r_3 is a root, distinct from both r_1 and r_2, of $f(x)=0$, then r_3 is a root of $R(x)=0$. In fact, the identity (10) holds when $x=r_3$ and then gives $0=(r_3-r_1)(r_3-r_2)R(r_3)$. Each of the first two factors is not zero, so that $R(r_3)=0$.

When two distinct roots r_1 and r_2 of $f(x)=0$ are known, it is usually more difficult to find its further roots r_3, etc., directly than to obtain them as the roots of the (doubly) depressed equation

(11) $$R(x)\equiv\frac{f(x)}{(x-r_1)(x-r_2)}=0$$

of degree $n-2$.

12. Factored Form of a Polynomial. Consider the polynomial (7) of degree n. When $n=2$, its factored form was found in § 6. For any n we shall prove the following generalization.

THEOREM 1. *If an equation* $f(x)=ax^n+\cdots=0$ *of degree* n *has* n *distinct roots* $r_1, \cdots, r_n$, *then* $f(x)$ *can be expressed in the factored form*

$$f(x)\equiv a(x-r_1)(x-r_2)\cdots(x-r_n). \tag{12}$$

The proof is an extension of the process which led us to identity (10). We saw that r_3 is a root of $R(x)=0$, so that

$$R(x)\equiv(x-r_3)S(x), \qquad S(x)\equiv ax^{n-3}+Ux^{n-4}+\cdots+W.$$

Insert this expression for $R(x)$ into (10). We get

$$f(x)\equiv(x-r_1)(x-r_2)(x-r_3)S(x).$$

If $n=3$, this is (12). If $n>3$, we take $x=r_4$ and see that $S(r_4)=$ since $r_4-r_1\neq 0$, etc. Thus

$$S(x)\equiv(x-r_4)(ax^{n-4}+\cdots).$$

Repetitions of this process evidently lead to (12).

EXAMPLE 1. Find a cubic equation having the roots 0, 1, -1.

Solution. By identity (12) one answer is $x(x-1)(x+1)=x^3-x=0$.

EXAMPLE 2. Solve $x^3-6x^2+11x-6=0$, given the root 3.

Solution. Here (9) with $r_1=3$ becomes $x^2-3x+2=0$, whose roots 1 and 2 are therefore the remaining roots of the cubic equation.

EXAMPLE 3. Solve $f(x)\equiv x^4+2x^3-7x^2-8x+12=0$, given the roots 1 and -2.

Solution. Here (11) is

$$\frac{f(x)}{;-2} \qquad ;-6=0, \text{ roots 2 and } -3.$$

PROBLEMS

1. Find a cubic equation having the roots 0, 1, 2.
2. Find a cubic equation with the roots -1, -2, -3.
3. Find a quartic equation having the roots ± 1 and ± 2.
4. Find a quartic equation having the roots 0, ± 1, 2.
5. Solve $x^3-7x^2+12x=0$.

Use synthetic division in Problems 6–14.

6. Solve $x^3+6x^2+11x+6=0$, given the root -2.

7. Solve $x^4+2x^3-7x^2-8x+12=0$, given the roots 2 and -3.

8. Solve $x^4+6x^3+13x^2+12x+4=0$, given the roots -1, -2.

9. Find the quotient of $f(x)=x^3+5x^2-2x-24$ by $x+4$, and then divide the quotient by $x+3$. What are the roots of $f(x)=0$?

10. Given that $x^4+2x^3-7x^2-8x+12=0$ has the roots 1 and -2, find the quadratic equation whose roots are the remaining two roots of the given equation, and find these roots. *Ans.* 2, -3.

11. If $x^4+2x^3-12x^2-10x+3=0$ has the roots -1 and 3, find the remaining two roots.

12. Solve $x^3+4x^2-47x-210=0$, given the roots 7 and -5.

13. Solve $x^4-3x^3+3x^2-3x+2=0$, given the roots 1 and 2.

14. Solve $x^3-27x+54=0$, given the roots 3 and -6.

15. Why is there a single answer to each of Problems 1–4 if the coefficient of the highest power of the unknown is taken to be unity?

16. What are further answers to Problems 1–4?

13. At Most n Roots.

We shall prove the following useful fact.

THEOREM 2. *An equation of degree* n *cannot have more than* n *distinct roots.*

Proof. Assume that the equation $f(x)\equiv ax^n+\cdots=0$ has $n+1$ distinct roots $r, r_1,\cdots, r_n$, and that $a\neq 0$. Then $f(x)$ has the factored form (12). Taking $x=r$ in that identity, we get

$$0=a(r-r_1)(r-r_2)\cdots(r-r_n).$$

By hypothesis, no factor on the right is zero. This contradiction shows that our assumption is false.

14. Identical Polynomials.

THEOREM 3. *If* $d_0x^n+d_1x^{n-1}+\cdots+d_n$ *has the value zero for more than* n *distinct values of* x, *it is identically zero* (*that is,* $d_0=0$, $d_1=0,\cdots, d_n=0$).

Proof. If $d_0\neq 0$, the equation $d_0x^n+\cdots+d_n=0$ has more than n distinct roots, contrary to Theorem 2. Hence $d_0=0$. Then if $d_1\neq 0$, the equation $d_1x^{n-1}+\cdots+d_n=0$ has more than n and hence more than $n-1$ distinct roots. This contradiction to the theorem cited gives $d_1=0$, etc

THEOREM 4. *If two polynomials in* x *of degree* n,

$$a_0x^n+a_1x^{n-1}+\cdots+a_n, \quad b_0x^n+b_1x^{n-1}+\cdots+b_n$$

are equal in value for more than n *distinct values of* x, *they are term by term identical* (*that is*, $a_0=b_0$, $a_1=b_1, \cdots, a_n=b_n$).

Proof. Write $d_0=a_0-b_0, \cdots, d_n=a_n-b_n$. Then a difference of the two polynomials has the form and properties in Theorem 3. But $d_0=0$ implies $a_0=b_0$, etc.

15. Multiple Roots. While the equation $x^2-4x+4=0$ has the single root 2, its factored form $(x-2)^2=0$ justifies our agreement that it has the root 2 counted twice, so that 2 is a double root. Similarly, the equation

$$7(x-4)(x-3)^2(x+2)^3(x-6)^4=0 \tag{13}$$

is said to have the simple root 4, double root 3, triple root -2, and four-fold root 6 (or root 6 of multiplicity 4). It evidently has no further root. If the multiplicity of a root exceeds 1, the root is called a *multiple* root. Thus any root is either a simple or a multiple root.

In general, R_1 is called an m_1-fold root or a root of *multiplicity* m_1 of $f(x)=0$ if and only if $f(x)$ has the factor $(x-R_1)^{m_1}$, but not the factor $(x-R_1)^{m_1+1}$. Then in

$$f(x)\equiv(x-R_1)^{m_1}\,q(x),$$

R_1 is not a root of $q(x)=0$. If R_2 is an m_2-fold root of $q(x)=0$, then

$$q(x)\equiv(x-R_2)^{m_2}\,h(x), \quad f(x)\equiv(x-R_1)^{m_1}(x-R_2)^{m_2}\,h(x),$$

where R_1 and R_2 are distinct and neither is a root of $h(x)=0$, while R_2 is an m_2-fold root of $f(x)=0$.

Conversely, let R_2 be an m_2-fold root of $f(x)=0$, and let $R_2\neq R_1$. The first identity shows that $q(R_2)=0$. Call m the multiplicity of the root R_2 of $q(x)=0$. It was just proved that R_2 is then an m-fold root of $f(x)=0$, so that $m=m_2$.

If $f(x)=0$ has an m_3-fold root R_3 which is distinct from R_1 and R_2, we see similarly that

$$f(x)\equiv(x-R_1)^{m_1}(x-R_2)^{m_2}(x-R_3)^{m_3}\,Q(x).$$

Proceeding in this manner, we obtain the following result.

THEOREM 5. *If an equation* $f(x)=ax^n+\cdots=0$ *of degree* n *has certain distinct roots* $R_1, \cdots, R_k$ *of multiplicities* $m_1, \cdots, m_k$, *and if* $m_1+\cdot\cdot$

, f(x) *can be expressed in the factored form*

$$f(x)\equiv a(x-R_1)^{m_1}(x-R_2)^{m_2}\cdots(x-R_k)^{m_k}. \tag{14}$$

We shall often write (14) in the form (12) with the understanding that $r_1, \cdots, r_n$ need not be distinct.

As in §13, the identity (14) implies

THEOREM 6. *An equation of degree* n *cannot have more than* n *roots provided a root of multiplicity* m *is counted* m *times.*

For example, equation (13) of degree 10 has no root other than 4, 3, −2, 6, while the sum of their multiplicities is $1+2+3+4=10$.

Usually multiple roots are best found by use of derivatives (§ 49).

PROBLEMS

Find the factored form of a quartic equation having

1. Double roots 2 and −2.
2. Triple root 3 and root 1.
3. Double root 4 and roots ±3.
4. Root 3 of multiplicity 4.
5. Describe the roots of $5(x-2)(x-4)^3(x-7)^4=0$.
6. What is the condition that $ax^2+bx+c=0$ shall have a double root?
7. Can a quartic equation have a double root and a triple root?

16. Relations between the Roots and the Coefficients. In § 6, we first found the sum and the product of the two roots of any quadratic equation and then deduced the factored form (4) of the equation. We shall here employ the reverse process for any equation of degree n.

For example, consider the case $n=3$. In Chapter V we shall learn how to solve any cubic equation $f(x)=0$ and hence find its roots r_1, r_2, r_3. Then

$$f(x)=x^3+C_1x^2+C_2x+C_3\equiv(x-r_1)(x-r_2)(x-r_3). \tag{15}$$

By actual multiplication this product is found to be

$$x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_1r_3+r_2r_3)x-r \tag{16}$$

Since this must be identical term by term with $f(x)$, we get

$$r_1+r_2+r_3=-C_1, \quad r_1r_2+r_1r_3+r_2r_3=C_2, \quad r_1r_2r_3=-C_3. \tag{17}$$

These formulas may be expressed in words as follows.

THEOREM 7. *For a cubic equation* $x^3+C_1x^2+C_2x+C_3=0$ *having unity as its leading coefficient, the sum of its roots is equal to the negative of the coefficient of* x^2, *the sum of the products of the roots two at a time is equal to the coefficient of* x, *and the product of the three roots is equal to the negative of the constant term.*

The fact that the product (15) has the expansion (16) is the case $n=3$ of the following general formula:

$$(18)\qquad (x-r_1)(x-r_2)\cdots(x-r_n)\equiv x^n-S_1x^{n-1}+S_2x^{n-2}-\cdots+(-1)^nS_n,$$

where $S_1, S_2, S_3, \cdots, S_n$ denote the left members of formulas (20) below, which are conveniently expressed in words in the later Theorem 8.

To prove this fact by mathematical induction, we assume that equation (18) holds for a fixed value of n and shall verify the equation obtained from (18) by replacing n by $n+1$. Since (18) was proved when $n=3$, it will therefore hold when $n=4$, and hence when $n=5$, etc. We multiply each member of equation (18) by $x-r_{n+1}$. The new second member is seen to be

$$x^{n+1}-(S_1+r_{n+1})x^n+(S_2+r_{n+1}S_1)x^{n-1}-(S_3+r_{n+1}S_2)x^{n-2}+\\ \cdots+(-1)^{n+1}r_{n+1}S_n.$$

The sums in parentheses are seen at once to be, respectively, the sum of $r_1,\cdots,r_n, r_{n+1}$, the sum of their products taken two at a time; the sum of their products taken three at a time; etc. Finally, $r_{n+1}S_n=r_1r_2\cdots r_nr_{n+1}$. This completes the proof that formula (18) holds also when n is replaced by $n+1$.

Consider an equation of the form

$$(19)\qquad f(x)=x^n+C_1x^{n-1}+\cdots+C_n=0$$

which has the roots $r_1,\cdots,r_n$, not necessarily distinct. By formula (14) and the remark below it, we see that the polynomial (19) is identically equal to the product in (18), and hence is term by term identical with the second member of (18). Hence we have proved the following relations.

$$(20)\qquad \begin{aligned} r_1+r_2+\cdots+r_n&=-C_1,\\ r_1r_2+r_1r_3+r_2r_3+\cdots+r_{n-1}r_n&=C_2,\\ r_1r_2r_3+r_1r_2r_4+\cdots+r_{n-2}r_{n-1}r_n&=-C_3,\\ \cdots\cdots\cdots\cdots\cdots&\\ r_1r_2\cdots r_{n-1}r_n&=(-1)^nC_n. \end{aligned}$$

These expressions are called the *elementary symmetric functions* of the roots $r_1, \cdots, r_n$. We have now proved the following generalization of Theorem 7.

THEOREM 8. *If an equation* (19) *of degree* n, *in which the coefficient of* x^{n} *is unity, has the roots* $\mathrm{r}_1, \cdots, \mathrm{r}_{\mathrm{n}}$, *not necessarily distinct, then relations* (20) *hold. In words, the sum of the* n *roots is equal to the negative of the coefficient of* $\mathrm{x}^{\mathrm{n}-1}$; *the sum of the products of the roots two at a time is equal to the coefficient of* $\mathrm{x}^{\mathrm{n}-2}$; *the sum of the products of the roots three at a time is equal to the negative of the coefficient of* $\mathrm{x}^{\mathrm{n}-3}$; *etc.*; *finally, the product of all the roots is equal to the constant term or its negative, according as* n *is even or odd.*

When the given equation is $c_0x^n + c_1x^{n-1} + \cdots = 0$, in which $c_0 \neq 0$, $c_0 \neq 1$, we divide its terms by c_0 and obtain (19), in which

$$C_1 = c_1/c_0,\ C_2 = c_2/c_0, \cdots,\ C_n = c_n/c_0.$$

When these values of the C's are inserted into (20), we obtain the desired relations between the roots and the coefficients. In the case $n=2$, these relations become formulas (3).

EXAMPLE 1. Solve $x^4+6x^3+13x^2+12x+4=0$, which has two double roots.

Solution. Denote the roots by s, s, t, t. Then the first two relations (20) become

$$2s+2t=-6, \qquad s^2+4st+t^2=13.$$

Subtract $(s+t)^2=9$ from the latter. We get $2st=4$. Hence s and t are the roots -1 and -2 of $y^2+3y+2=0$.

EXAMPLE 2. Solve $x^3-7x^2+36=0$, given that one root is double another.

Solution. Denote the roots by $s, t, 2t$. Then the first two relations (20) become

$$s+3t=7, \qquad 3st+2t^2=0.$$

Since no root is zero, $t \neq 0$, $3s+2t=0$. The two linear equations have the unique set of solutions $s=-2$, $t=3$.

PROBLEMS

Without using linear factors in Problems 1–4,

1. Find a cubic equation having the roots 1, 2, 3. *Ans.* $x^3-6x^2+11x-6=0$.
2. Find a cubic equation having the roots 0, 2, -2.
3. Find a quartic equation having the double roots 2 and -2. *Ans.* $x^4-8x^2+16=0$.
4. Find a quartic equation having the simple root 1 and triple root 2.
5. Solve $x^4+14x^3+73x^2+168x+144=0$, which has two double roots.
6. Solve $9x^4-42x^3+13x^2+84x+36=0$, which has two double roots.

7. Solve $x^3-27x^2+242x-720=0$, one root being half the sum of the remaining two. *Ans.* 8, 9, 10.

8. Solve $x^3-14x^2+61x-84=0$, one root being the sum of the others.

9. Solve $x^3+7x^2-6x-72=0$, two roots being in the ratio of 3 to 2. *Ans.* -6, -4, 3.

10. Solve $2x^3-7x^2+4x+3=0$, given that the sum of two roots is 2.

11. Solve $x^3-28x-48=0$, given that two roots differ by 2.

12. Solve $x^3-9x^2+23x-15=0$, given that one root is triple another. *Ans.* 1, 3, 5.

Given that one root is the negative of another in Problems 13–17,

13. Solve $4x^3-12x^2-25x+75=0$.

14. Solve $4x^3-16x^2-9x+36=0$. *Ans.* $\frac{3}{2}, \frac{-3}{2}$, 4.

15. $x^3+px^2+qx+r=0$ must have $r=pq$.

16. Solve $x^3-3x^2-16x+48=0$.

17. Solve $3x^3-x^2-15x+5=0$. *Ans.* $\frac{1}{3}$, $\pm\sqrt{5}$.

18. Solve $x^4-12x^3+48x^2-80x+48=0$, which has a triple root.

19. Solve $x^4+6x^3+12x^2+10x+3=0$, which has a triple root. *Ans.* -1, -1, -1, -3.

20. Solve $x^3+7x^2-21x-27=0$, whose roots are in geometric progression (G. P.), with a common ratio r (say m/r, m, mr).

21. Solve $x^3-14x^2-84x+216=0$ with roots in G. P. *Ans.* 2, -6, 18.

22. Solve $x^3-3x^2-13x+15=0$, whose roots are in arithmetical progression (A. P.), with a common difference d (say $m-d$, m, $m+d$). *Ans.* -3, 1, 5.

23. Solve $x^4-2x^3-21x^2+22x+40=0$, whose roots are in A. P. (Denote them by $c-3b$, $c-b$, $c+b$, $c+3b$, with the common difference $2b$.) *Ans.* 5, 2, -1, -4.

24. Solve $x^3+6x^2-52x-120=0$, with roots in A. P.

25. Solve $x^4+4x^3-84x^2-176x+640=0$, with roots in A. P.

26. Solve $x^3+9x^2+26x+24=0$, with roots in A. P.

27. Find a necessary and sufficient condition that the roots, taken in some order, of $x^3+px^2+qx+r=0$ shall be in G. P. *Ans.* $p^3r=q^3$.

Given that r and s are the roots of $x^2-px+q=0$ in Problems 28–32, find an equation whose roots are

28. r^2, s^2. *Ans.* $y^2-(p^2-2q)y+q^2=0$.

29. r^3, s^3. *Ans.* $y^2-(p^3-3pq)y+q^3=0$.

30. r^2/s, s^2/r. *Ans.* $y^2-y(p^3-3pq)/q+q=0$.

31. r^3s, rs^3. *Ans.* $y^2-q(p^2-2q)y+q^4=0$.

32. $r+1/s$, $s+1/r$. *Ans.* $y^2-(p+p/q)y+2+q+1/q=0$.

17. Imaginary Roots Occur in Pairs. The two roots of a real quadratic equation whose discriminant is negative are conjugate imaginaries (§ 6). This fact illustrates the following useful result.

Theorem 9. *If an algebraic equation with real coefficients has the root* a+bi, *where* a *and* b *are real and* b≠0, *it has also the root* a−bi.

Proof. By Problem 12 of § 2, the conjugate of the product of two complex numbers is equal to the product of their conjugates. This implies that the conjugate of x^n is $(\bar{x})^n$, if $\bar{x}$ denotes the conjugate of x. If c is real, $\bar{c}=c$. By Problem 11 of § 2, the conjugate of a sum is equal to the sum of the conjugates. These facts show that, if $f(x)$ is a polynomial with real coefficients, its conjugate is $f(\bar{x})$. In particular, if $f(a+bi)=0$, then $f(a-bi)=0$.

THEOREM 10. *If a real algebraic equation has an imaginary root* r *of multiplicity* m, *the conjugate imaginary of* r *is a root of multiplicity* m. *If an* m*-fold root is counted* m *times, the total number of imaginary roots is even.*

Proof. By hypothesis, $f(x)$ is divisible by $(x-r)^m$, but not by $(x-r)^{m+1}$. We saw that the conjugate of $f(x)$ is $f(\bar{x})$. Hence the latter has the factor $(\bar{x}-\bar{r})^m$. Changing the notation, we see that $f(y)$ has the factor $(y-\bar{r})^n$, where $n\geqq m$. But if $n>m$, we repeat our argument and see that $f(x)$ has the factor $(x-r)^n$, contrary to hypothesis.

EXAMPLE. Solve $x^4-7x^2+20x+14=0$, one root being $2-\sqrt{3}i$.

Solution. Both of $2\pm\sqrt{3}i$ are roots. They are the roots of $x^2-4x+7=0$. Dividing the quartic function by this quadratic function, we get the quotient x^2+4x+2, which vanishes for $x=-2\pm\sqrt{2}$.

PROBLEMS

1. Solve $x^3+6x^2-24x+160=0$, one root being $2-2\sqrt{-3}$.

2. Solve $x^3-3x^2-6x-20=0$, one root being $-1+\sqrt{-3}$. *Ans.* 5, $-1\pm\sqrt{-3}$.

3. Solve $x^4-4x^2+8x-4=0$, one root being $1+i$.

4. Solve $x^4-4x^3+5x^2-2x-2=0$, one root being $1+i$. *Ans.* $1\pm i$, $1\pm\sqrt{2}$.

5. Find a real cubic equation two of whose roots are 2 and $3-\sqrt{-2}$.

6. Find a real cubic equation two of whose roots are 1 and $3+2i$. *Ans.* $x^3-7x^2+19x-13=0$.

7. If a real cubic equation $x^3+\cdots-20=0$ has the root $3+i$, what are its remaining roots?

8. If a real cubic equation $x^3-6x^2+\cdots=0$ has the root $1+\sqrt{-5}$, what are the remaining roots? *Ans.* 4, $1-\sqrt{-5}$.

9. Find a real quartic equation having the double root $2-i$.

10. Granted that a real cubic equation has the root 3 and no real root different from 3, does it have imaginary roots?

11. Granted that a real quartic equation has the roots $2\pm3i$ and no imaginary root different from them, does it have two real roots? *Ans.* Not necessarily.

12. The equation $x^3-(8+i)x^2+(19+7i)x-12-12i=0$ has the root $1+i$. Does it have the root $1-i$?

13. If $x^3+px+q=0$ is a real equation with the imaginary root $a+bi$, it has the real root $-2a$.

The problem to find the imaginary roots when no root is known is treated in § 99.

THEOREM 11. *If the equation* $\mathrm{f(x)}=0$ *with rational* coefficients has the root* $\mathrm{a}+\sqrt{\mathrm{b}}$, *where* a *and* b *are rational, but* b *is not the square of a rational number, the equation has the root* $\mathrm{a}-\sqrt{\mathrm{b}}$.

Proof. Divide $f(x)$ by,

(21) $$-b=(x-a-$$

until we reach a remainder $rx+s$ whose degree in x is less than the degree of the divisor. Since the coefficients of the dividend $f(x)$ and divisor (21) are all rational numbers, the coefficients of the quotient $q(x)$ and remainder $rx+s$ are rational. As in § 9, we have

$$f(x)\equiv(x^2-2ax+a^2-b)q(x)+rx+s,$$

identically in x. This identity is true in particular when $x=a+\sqrt{b}$, so that $0=r(a+\sqrt{b})+s$. If $r\neq 0$, this gives $\sqrt{b}=(-s-ra)/r$, which contradicts the assumption that b is not the square of a rational number. Hence $r=0$, so that $s=0$. This proves that $f(x)$ is exactly divisible by the function (21) and hence by $x-a+\sqrt{b}$. In other words, $f(x)=0$ has the root $a-\sqrt{b}$.

PROBLEMS

1. Solve $x^3-3x^2-5x+7=0$, given the root $1-\sqrt{8}$.

2. Solve $x^4-13x^2+4x+2=0$, given the root $2-\sqrt{2}$. *Ans.* $2\pm\sqrt{2}$, $-2\pm\sqrt{3}$.

3. If an equation $x^3-6x^2+\cdots=0$ has rational coefficients and has the root $1-\sqrt{5}$, what are the remaining roots?

4. If an equation $x^3+\cdots+28=0$ with rational coefficients has the root $3-\sqrt{2}$, what are its remaining roots?

5. Given that $x^4-2x^3-5x^2-6x+2=0$ has the root $2-\sqrt{3}$, use the sum and the product of the four roots to obtain, without division, the quadratic equation satisfied by the imaginary roots. *Ans.* $x^2+2x+2=0$.

6. Solve $x^4-3x^2+10x-6=0$, given two roots $1-\sqrt{-2}$ and $-1+\sqrt{3}$.

7. Extend Theorem 11 to a root $a+\sqrt{b}$ of multiplicity m. Hint: Apply that theorem to the quotient $q(x)$.

* All positive or negative whole numbers or fractions, and zero, are called *rational* numbers.

CHAPTER III

AND RATIONAL ROOTS; UPPER LIMIT TO REAL ROOTS

18. Integral Roots. The positive and negative whole numbers are called *integers.*

THEOREM 1. *For an equation all of whose coefficients are integers, any integral root is an exact divisor of the constant term.*

Proof. If x is an integer such that

(1) $$ax^n+\cdots+jx^2+kx+l=0 \quad (a,\cdots, k, l \text{ integers}),$$

then, by transposing all terms before l, we get

$$xq=l, \quad q=-ax^{n-1}-\cdots-k.$$

Evidently q is an integer. Since the product of the two integers x and q gives l, x is called an exact divisor of l.

EXAMPLE 1. Find all integral roots of $x^3+x^2-3x+9=0$.

Solution. The only exact divisors of the constant term 9 are ±1, ±3, ±9. By trial, 1, -1, 3 are not roots. We may verify that -3 is a root by synthetic division:

$$\begin{array}{rrrr|l} 1 & 1 & -3 & 9 & \underline{-3} \\ & -3 & 6 & -9 & \\ \hline 1 & -2 & 3 & 0 & \end{array}$$

In the bottom line, the entry 0 shows that -3 is a root, while the earlier entries are the coefficients of the quotient x^2-2x+3 obtained when we divide our cubic function by $x+3$. The roots other than -3 are the roots of $x^2-2x+3=0$. Its constant term 3 is not divisible by either ±9 (which are the original divisors not yet examined). Hence -3 is the only integral root. Or, we may solve $x^2-2x+3=0$ and find the imaginary roots $1\pm\sqrt{2}i$ of the quadratic and cubic equations.

When the constant term has numerous exact divisors, we should use the method of § 20 unless we notice a device like that in Ex. 2 which simplifies the application of our present theorem.

EXAMPLE 2. Find all integral roots of

$$y^3+12y^2-32y-256=0,$$

whose constant term is -2^8 and thus has 18 exact divisors.

Solution. Since all of the terms except y^3 are divisible by 2, any integral root y must be divisible by 2. Therefore $y=2z$, where z is an integer. Replacing y by $2z$ in the given equation, and removing the factor 2^3, we get

All the terms except z^3 are divisible by 2. Hence $z=2x$, where x is an integer. Replac ing z by $2x$ and removing the factor 2^3, we get

As in Ex. 1, it is readily found to have the roots -1, $-1\pm\sqrt{5}$. Since $y=4x$, their products by 4 give all the roots of the proposed equation in y, so that -4 is the only integral root of the latter.

PROBLEMS

1. If each coefficient is positive or zero, or if each is negative or zero, there is no positive root. If this becomes true after x is replaced by $-x$, there is no negative root. These facts shorten the work in several of the later problems.

Find all the integral roots of

2. $x^3+16x^2+52x+48=0$.
3. $x^3-2x^2-22x-12=0$.
4. $x^3-3x+1=0$.
5. $x^3+x^2-2x-1=0$.
6. $x^3-10x^2+18x-16=0$.
7. x^3-
8. $x^4-4x^3-8x+32=0$.
9. x^4-
10. $x^4+2x^2+4x+8=0$.
11.

12. Why may we not deduce Theorem 1 from the fact that the product of all n roots is $\pm l/a$?

13. The root 2 of $x^2-\frac{1}{2}x-3=0$ is not a divisor of -3. Explain.

19. Upper Limit to the Real Roots. Any number which exceeds all the real roots is called an *upper limit* to the real roots. If the coefficients of an equation are all of like sign, there is evidently no positive root. We shall here exclude such an equation since we already know an upper limit zero to its real roots. All remaining real equations $f(x)=0$ have at least one negative coefficient and at least one positive coefficient. In case the coefficient of the highest power of x is negative, we replace our equation $f(x)=0$ by $-f(x)=0$.

By the numerical value of a negative number -5 or $-p$ is meant 5 or p, respectively. Thus -5 is numerically greater than 4.

THEOREM 2. *If, in a real equation*

$$f(x)=c_0x^n+c_1x^{n-1}+\cdots+c_n=0 \quad (c_0>0),$$

the first negative coefficient is preceded by k *coefficients which are positive or zero, and if* G *denotes the greatest of the numerical values of the negative coefficients, then each real root is less than* $1+\sqrt[k]{G/c_0}$.

For example, in $x^5+4x^4-7x^2-40x+1=0$, we have $G=40$ and $k=3$ since we must supply the coefficient zero to the missing power x^3. Thus the theorem asserts that each root is less than $1+\sqrt[3]{40}$ and therefore less than 4.42. Hence 4.42 is an upper limit to the roots.

Proof. For positive values of x, $f(x)$ will be reduced in value or remain unchanged if we omit the terms $c_1x^{n-1}, \cdots, c_{k-1}x^{n-k+1}$ (which are positive or zero), and if we change each later coefficient $c_k, \cdots, c_n$ to $-G$. Hence

$$f(x) \geqq c_0x^n-G(x^{n-k}+x^{n-k-1}+\cdots+x+1).$$

But, by Problem 15 of § 9,

$$x^{n-k}+\cdots+x+1 \equiv \frac{x^{n-k+1}-1}{x-1},$$

if $x\neq 1$. Furthermore,

$$c_0x^n-G\left(\frac{x^{n-k+1}-1}{x-1}\right) \equiv \frac{x^{n-k+1}\{c_0x^{k-1}(x-1)-G\}+G}{x-1}.$$

Hence, if $x>1$,

$$f(x)>\frac{x^{n-k+1}\{c_0x^{k-1}(x-1)-G\}}{x-1},$$

$$f(x)>\frac{x^{n-k+1}\{c_0(x-1)^k-G\}}{x-1}.$$

Thus, for $x>1$, $f(x)>0$ and x is not a root if $c_0(x-1)^k-G\geqq 0$, which is true if $x\geqq 1+\sqrt[k]{G/c_0}$.

THEOREM 3. *If, in a real algebraic equation in which the coefficient of the highest power of the unknown is positive, the numerical value of each negative coefficient be divided by the sum of all the positive coefficients which precede it, the greatest quotient so obtained increased by unity is an upper limit to the real roots.*

For example, in $x^5+4x^4-7x^2-40x+1=0$, the quotients are $7/(1+4)$ and $40/5$, so that Theorem 3 asserts that $1+8$ or 9 is an upper limit to the roots. Theorem 2 gave the better upper limit 4.42. But for $x^3+8x^2-9x+c^2=0$, Theorem 2 gives the upper limit 4, while Theorem 3 gives the better upper limit 2.

We shall first give the proof for

$$f(x)=px^4-qx^3+rx^2-sx+t=0,$$

in which $p, q, \cdots, t$ are all positive. Write d for $x-1$. Then

$$x^4 \equiv d(x^3+x^2+x+1)+1, \qquad x^2 \equiv d(x+1)+1.$$

Replacing x^4 and x^2 by these expressions, we see that

$$\begin{aligned} f(x) \equiv pdx^3 &+ pdx^2 + pdx + pd + p \\ -qx^3 &\qquad\quad + rdx + rd + r \\ &\qquad\quad - sx \qquad + t. \end{aligned}$$

Let $x>1$, so that $d>0$. Then negative terms occur only in the first and third columns. The sum of the terms in the first column will be $\geqq 0$ if $pd-q \geqq 0$. Likewise for the third column if $(p+r)d-s \geqq 0$. But $d=x-1$. Hence $f(x)>0$ (and x is not a root) if

$$x \geqq 1+\frac{q}{p}, \qquad x \geqq 1+\frac{s}{p+r}.$$

This evidently proves Theorem 3 for the present equation.

To extend this method of proof to the general case

$$f(x)=a_n x^n+\cdots+a_1 x+a_0 \quad (a_n>0),$$

we require suitable general notations. Let the negative coefficients in order be $a_{k_1}, \cdots, a_{k_t}$, so that $k_1>k_2>\cdots>k_t$. For each positive integer m which is $\leqq n$ and distinct from $k_1, \cdots, k_t$, we replace x^m by the equal value

$$d(x^{m-1}+x^{m-2}+\cdots+x+1)+1,$$

where $d=x-1$. Let $F(x)$ denote the polynomial in x, with coefficients involving d, which is obtained from $f(x)$ by these replacements. Let $x>1$, so that d is positive. Thus the terms $a_{k_i}x^{k_i}$ having $i=1, \cdots, t$ are the only negative quantities occurring in $F(x)$. If $k_i>0$, the terms of $F(x)$ which involve explicitly the power x^{k_i} are $a_{k_i}x^{k_i}$ and the $a_m dx^{k_i}$ for the various positive coefficients a_m which precede a_{k_i}. The sum of these terms will be $\geqq 0$ if $a_{k_i}+d\Sigma a_m \geqq 0$, and hence if

$$x \geqq 1+\frac{-a_{k_i}}{\Sigma a_m}.$$

There is an additional case if $k_t=0$, i.e., if a_0 is negative. Then the terms of $F(x)$ not involving x explicitly are a_0 and the $a_m(d+1)$ for the various positive coefficients a_m. Their sum, $a_0+x\Sigma a_m$, will be >0 if

$$x>\frac{-a_0}{\Sigma a_m},$$

which is true if

$$x\geqq 1+\frac{-a_0}{\Sigma a_m}.$$

PROBLEMS

Apply both Theorems 2 and 3 to find an upper limit to the roots of

1. $4x^5-8x^4+22x^3+98x^2-73x+5=0$. *Ans.* $19\frac{1}{4}$, 3.
2. $x^4-10x^3+28x^2-64x+16=0$.
3. $x^7+3x^6-4x^5+5x^4-6x^3-7x^2-8=0$. *Ans.* 2.
4. $x^3-20x^2+164x-400=0$.
5. $2x^3-7x^2+10x-6=0$.
6. $x^7+2x^5+4x^4-8x^2-32=0$. *Ans.* 3.
7. $x^8-5x^6+7x^4-8x^2+1=0$.
8. $x^4-41x^2+400=0$.
9. $x^4-8x^3+18x^2-16x+5=0$.
10. $2x^3-5x^2+x+10=0$.
11. A lower limit to the negative roots of $f(x)=0$ may be found by applying our theorems to $f(-x)=0$, which is the equation derived from $f(x)=0$ by replacing x by $-x$. Find a lower limit to the negative roots in Problem 3. *Ans.* -7.

20. Best Method for Integral Roots.

THEOREM 4. *If* f(x) = 0 *is an algebraic equation all of whose coefficients are integers, an integral divisor* d *of the constant term is not a root if an integer* m *can be found such that* d − m *is not a divisor of* f(m).

Proof. If d is a root of $f(x)=0$, then

$$f(x)\equiv(x-d)Q(x),$$

where $Q(x)$ is a polynomial having integral coefficients (§ 10). Hence $f(m)=(m-d)q$, where q is the integer $Q(m)$. This result contradicts the hypothesis that $d-m$ is not a divisor of $f(m)$. Hence our assumption that d is a root of $f(x)=0$ has led to a contradiction.

EXAMPLE. Find all integral roots of

$$f(x)=x^3-$$

whose constant term has 30 divisors.

Solution. By either Theorem 2 or Theorem 3, 21 is an upper limit to the roots. Evidently there is no negative root. The positive divisors less than 21 of $400=2^4 5^2$ are $d=1, 2, 4, 8, 16, 5, 10, 20$. First, take $m=1$ and note that $f(1)=-255=-3\cdot5\cdot17$. The corresponding values of $d-1$ are 0, 1, 3, 7, 15, 4, 9, 19; of these 7, 4, 9, 19 are not divisors of $f(1)$, so that $d=8, 5, 10$, and 20 are not roots. Next, take $m=2$ and note that $f(2)=-144$ is not divisible by $16-2=14$. Hence 16 is not a root. Incidentally, $d=1$ and $d=2$ were excluded since $f(d)\neq0$. There remains only $d=4$, which is a root.

PROBLEMS

Find by this best method all the integral roots of

1. $x^4+8x^3-57x^2-648x-1944=0$.
2. $x^3-9x^2-24x+216=0$, *Ans.* 9.
3. $x^3-8x^2-104x-384=0$.
4. $x^4-23x^3+187x^2-653x+936=0$, *Ans.* 8, 9.
5. $x^4+4x^3-75x^2-324x-486=0$.
6. $x^5+47x^4+423x^3+140x^2+1213x-420=0$, *Ans.* -12, -35.
7. $x^5-14x^4-3x^3+46x^2-14x-588=0$.
8. $x^3-48x+64=0$.
9. $x^3+4x^2-32x-64=0$, *Ans.* None.

21. Rational Roots.

THEOREM 5. *If an equation with integral coefficients*

$$ax^m+bx^{m-1}+cx^{m-2}+\cdots+kx+l=0 \qquad (2)$$

has the rational root n/d, *where* n *and* d *are integers having no common factor* >1, *then* n *is an exact divisor of the constant term* l, *and* d *is an exact divisor of the leading coefficient* a.

Proof. Insert the value n/d of x and multiply all terms of the resulting equation by d^m. We get

$$an^m+bn^{m-1}d+\cdots+knd^{m-1}+ld^m=0.$$

Since n divides all the terms preceding the final term, n divides that term. But n has no divisor >1 in common with d^m. Hence n divides l. Similarly d divides all terms after the first, and has no factor >1 in common with n^m; hence d divides a.

EXAMPLE. Find all rational roots of

Solution. By Theorem 5, the denominator of any rational root x is a divisor of 2. Hence $y=2x$ is an integer. To avoid fractions we multiply all terms of our equation by 4 before making the substitution y for $2x$. We get

$$y^3-7y^2+20y-24=0.$$

The only integral root is found to be $y=3$. Hence $x=3/2$ is the only rational root of the proposed equation.

Consider the case $a=1$ of Theorem 5. Then its divisor d is ± 1, so that any rational root n/d is an integer $\pm n$. This proves the following important fact.

THEOREM 6. *Consider an equation with integral coefficients such that the coefficient of the highest power is unity. Then every rational root is an integer.*

Given any equation with integral coefficients

$$Ay^n+By^{n-1}+Cy^{n-2}+\cdots+Ky+L=0,$$

we can readily transform it into an equation of the type defined in Theorem 6. We have only to multiply each term by A^{n-1} and write x for Ay; we get

$$x^n+Bx^{n-1}+CAx^{n-2}+\cdots+KA^{n-2}x+A^{n-1}L=0, \tag{3}$$

having integral coefficients and unity as the coefficient of x^n. By Theorem 6, each rational root x is an integer. Thus we need only find all the integral roots x and divide them by A to obtain all the rational roots y of the given equation.

Frequently it is sufficient, as in the following example, to set $ky=x$, where k is a integer less than A. We must choose k so that the coefficients of the resulting equation are all integers. They will be less than the corresponding coefficients of equation (3) if $k<A$.

EXAMPLE. Find the rational roots of $96y^3-16y^2-6y+1=0$.

Solution. Since $96=2^2\cdot 3\cdot 2^3$, the least multiple of 96 which is a cube is evidently $2\cdot 3^2\cdot 96=2^3\cdot 3^3\cdot 2^3$. Hence we multiply the terms of the given equation by $2\cdot 3^2$ and set $2^2\cdot 3y=x$. We get $x^3-2x^2-9x+18=0$. Its integral roots are found to be $x=2, 3, -3$. Hence the answers are $y=1/6, 1/4, -1/4$.

PROBLEMS

Find all the rational roots of

1. $3y^4-40y^3+130y^2-120y+27=0$, *Ans.* 1, 3, 9, 1/3.
2. $6y^3+7y^2-9y+2=0$.
3. $2y^3-y^2-4y+2=0$, *Ans.* 1/2.
4. $2y^3+y^2-2y-6=0$.
5. $3y^3-2y^2+9y-6=0$, *Ans.* 2/3.
6. $16y^3-4y^2-4y+1=0$.
7. $108y^3-270y^2-42y+1=0$, *Ans.* $-1/6$.
8. $32y^3-6y+1=0$.
9. $24y^3-2y^2-5y+1=0$, *Ans.* 1/4, 1/3, $-1/2$.
10. $x^3-3x-1=0$.
11. $x^3-x^2-2x+1=0$.

22. Other Methods for Integral and Rational Roots. In § 18, we transposed all terms before l and proved that an integral root x of equation (1) is an exact divisor of l. Similarly, by transposing all terms before $kx+l$, we see that $kx+l$ must be divisible by x^2 and hence $k+l/x$ must be divisible by x. Transposing all but the last three terms of (1), we see that their sum must be divisible by x^3, so that $J=j+(k+l/x)/x$ must be divisible by x; etc.

For example, 3 is not a root of

$$ax^n+\cdot \text{[illegible]},$$

although 3 is a divisor of 15 and of $4+15/3=9$, since 3 is not a divisor of $J=2+9/3=5$.

Since this method of Newton's requires the separate examination of all the divisors of l, it is usually much longer than Theorem 4.

There is a similar method of testing equation (2) for a fractional root n/d in its lowest terms. Then d must divide the integers

$$a,\quad \frac{an}{d}+b,\quad \frac{an^2}{d^2}+\frac{bn}{d}+c, \cdots$$

For example, in testing $96x^3-16x^2-6x+1=0$ for the root 1/3 by synthetic division

$$\begin{array}{rrrr|l} 96 & -16 & -6 & 1 & \underline{1/3} \\ & 32 & & & \\ \hline 96 & 16 & & & \end{array}$$

the next product $\frac{1}{3}\cdot 16$ is not an integer. Without proceeding further, we conclude that 1/3 is not a root.

We may also extend § 20 to rational roots. If n/d is a fractional root in its lowest terms, then $f(x)\equiv(x-n/d)Q(x)$, where $Q(x)$ has integral coefficients by the preceding discussion. Replacing x by an integer m, we see that d divides $Q(m)$ and then that $dm-n$ divides $f(m)$. For example, let $f(x)=96x^3-16x^2-6x+1$, $m=1$. When $n=1$, $d=3$, $dm-n=2$ does not divide $f(1)=75$, so that 1/3 is not a root. Similarly, $-1/6$ is not a root since $6+1=7$ does not divide 75.

CHAPTER IV

Impossibility of the Trisection of an Angle or Construction of Regular Polygons of Seven and Nine Sides by Ruler and Compasses

23. Impossible Constructions. Elementary geometries show how to bisect any angle, but not how to trisect it. They show how to construct regular polygons of 3, 4, 5, 6, 8, or 10 sides, but not regular polygons of 7 or 9 sides. Why do geometries fail to give constructions for the three problems omitted? The answer is that it is impossible to trisect all angles by ruler and compasses, and impossible to construct a regular polygon of 7 or 9 sides by the same tools. By ruler is meant a straight edge, not graduated.

Why do geometries fail to prove that these constructions are impossible? The answer is that the proof is beyond the scope of elementary geometry, since it requires the use of the theory of equations.

Why for centuries has there been an annual crop of angle-trisectors? The answer is not easy. Some angle-trisectors have not heard of the fact that there are proofs (by the theory of groups or by the more elementary method to be given here) which show that it is absolutely impossible to trisect all* angles by ruler and compasses. Others have heard of such proofs, but ignore them. Often such a person regards "impossible" as meaning merely that mathematicians have not to-date succeeded in finding a construction, whereas he may have more luck.

If a reader is interested in this problem of trisection, but not in the earlier chapters of this book, he can read in a few moments the proofs of the three simple facts required for our discussion of trisection. These are Theorems 1 and 6 of Chapter III, about integral and rational roots, and Theorem 7 of Chapter II, which gives the sum of the roots of a cubic equation. Moreover, he may replace the literal cubic equation (4) by the numerical equation (2).

* We can trisect special angles like 180° since we can construct an equilateral triangle, and each of its angles is 60°.

24. Problem of the Trisection of an Angle. Let A be the given angle which is to be trisected (if possible). Choose a convenient unit of length. On one arm of angle A, with vertex O, mark the point P so that the length of OP is unity. From P draw a line perpendicular to the other arm of A, produced if necessary. In Fig 5, $\cos A$ is the length of OQ. In Fig. 6, $\cos A$ is the negative of the length of OR. Likewise by analogous figures if $180° < A \leqq 360°$.

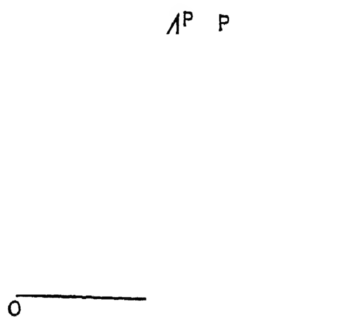


FIG. 5 FIG. 6

If it be possible to trisect angle A, i.e., construct angle $\frac{1}{3}A$ with ruler and compasses, the above discussion (with A replaced by shows that we could construct a line whose length is the positive value of $\pm\cos\frac{1}{3}A$.

It is proved in trigonometry that $\cos 3B = 4\cos^3 B - 3\cos B$ for every angle B. Take $B = \frac{1}{3}A$. We get

$$\cos A \quad \cos^3 \quad -3\cos$$

Multiply each term by 2 and write x for $2\cos\frac{1}{3}A$. Then

(1) $$x^3 - 3x -$$

Hence we have proved the first part of Lemma 1 stated at the end of this section.

For the present we shall be content if we can prove that angle 60° cannot be trisected with ruler and compasses. For $A = 60°$, triangle OQP in Fig. 5 is half of an equilateral triangle, so that the length of OQ is 1/2. Thus $\cos 60° = 1/2$ and equation (1) becomes

(2) $$x^3 - 3x - 1 = 0.$$

By Theorems 6 and 1 of Chapter III, any rational root of equation (2) is an integer, which is an exact divisor of the constant term. The divisors of -1 are $+1$ and -1. By trial, neither $+1$ nor -1 is a root of (2). Hence equation (2) has no rational root. This completes the proof of the following fact.

LEMMA 1. *Let a unit of length be given. If it be possible to trisect angle* A, *we could construct with ruler and compasses a line of length* x_1 *or* $-x_1$, *where* x_1 *is one of the roots of equation* (1). *If* $A=60°$, *the latter becomes equation* (2), *which has no rational root.*

25. Condition That a Proposed Construction Be Possible For example, it may be proposed to construct a line of given length ($\pm x_1$ in Lemma 1). In general, suppose that a proposed construction is possible with ruler and compasses. The straight lines and circles drawn in making the construction are located by means of points either initially given or obtained as the intersections of two straight lines, a straight line and a circle, or two circles. Since the axes of coordinates are at our choice, we may assume that the y-axis is not parallel to any of the straight lines employed in the construction. Then the equation of any one of our lines is

$$y=mx+b, \tag{3}$$

and not $x=c$. Let $y=m'x+b'$ be the equation of another of our lines which intersects (3). The coordinates of their point of intersection are

$$x=\frac{b'-b}{m-m'}, \qquad y=\frac{mb'-m'b}{m-m'},$$

which are rational functions of the coefficients of the equations of the two lines.

Suppose that a line (3) intersects the circle

with the center (c, d) and radius r. To find the coordinates of the points of intersection, we eliminate y between the equations and obtain a quadratic equation for x. Thus x (and hence also $mx+b$ or y) involves no new irrationality other than a real square root.

Finally, the intersections of two circles are given by the intersections of one of them with their common chord, so that this case reduces to the preceding one.

When this general discussion is applied to the example mentioned, it leads to the following result, which is sufficient for our present purposes.

LEMMA 2. *Let* x_1 *be a root of a cubic equation having rational coefficients. Let a unit of length be given. If it be possible to construct with ruler and*

compasses a line of length x_1 *or* $-x_1$, *then* x_1 *can be obtained by a finite number of rational operations (addition, subtraction, multiplication, division) and extractions of real square roots, performed on rational numbers or on numbers derived from rational numbers by such operations.*

26. Cubic Equations with a Constructible Root. After a brief interruption we shall prove our third lemma.

LEMMA 3. *Let a unit of length be given. If* (i) *a line of length* x_1 *or* $-x_1$ *can be constructed with ruler and compasses and if* (ii) x_1 *is a root of*

$$x^3+px^2+qx+r=0 \quad (p, q, r \text{ rational}), \tag{4}$$

then at least one root of (4) *is rational.*

This implies the following result. *If* (ii) *holds, and if no root of equation* (4) *is rational, then a line of length* x_1 *or* $-x_1$ *cannot be constructed.* For, if we deny this conclusion, we have (*i*) as well as (*ii*), so that Lemma 3 shows that at least one root of (4) is rational. But the latter contradicts our present second hypothesis. Thus the denial has led to a contradiction.

The last result stated in italics may be restated as follows.

THEOREM 1. *It is impossible to construct with ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root, when the unit of length is given.*

From this theorem and Lemma 1 it follows at once that it is impossible to trisect angle 60° with ruler and compasses.

Nor can we so trisect angle 120°. For, if that were possible, we could construct angle 40° and by bisection construct angle 20° and hence trisect angle 60°. The interesting question as to which angles can be trisected and which cannot, when the cosine of the angle is a rational number, is answered in § 30 (see Problems 1, 2, 3, which show that a very small percentage of such angles can be trisected).

27. Proof of Lemma 3. In case x_1 itself is rational there is nothing to prove. Hence let x_1 be irrational. Since the hypotheses in Lemma 3 are the same as those in Lemma 2, the latter shows that the irrational number x_1 involves one or more real square roots, but no irrationality other than real square roots.

There may be superimposed radicals as in the length

$$\tag{5} \tfrac{1}{2}\sqrt{10-2\sqrt{5}}$$

of a side of a regular pentagon inscribed in a circle of radius unity.* In case such a two-story radical is not expressible as a rational function, with rational coefficients, of a finite number of square roots of positive rational numbers, it is said to be a radical of *order* 2. In general, an n-story radical is said to be of order n if it is not expressible as a rational function, with rational coefficients, of radicals each with fewer than n superimposed radicals, the innermost ones affecting positive rational numbers.

We agree to simplify x_1 by making all possible replacements of certain types that are sufficiently illustrated by the following numerical examples. If x_1 involves $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{15}$, we agree to replace $\sqrt{15}$ by $\sqrt{3}\cdot\sqrt{5}$, and to replace $(\sqrt{5})^2$ by 5. If $x_1=s-7t$, where s is given by (5) and

$$t=\tfrac{1}{2}\sqrt{10+2\sqrt{5}},$$

so that $st=\sqrt{5}$, we agree to write x_1 in the form $s-7\sqrt{5}/s$, which involves a single radical of order 2 and no new radical of lower order. Finally, we agree to replace $\sqrt{4-2\sqrt{3}}$ by its simpler form $\sqrt{3}-1$.

After all possible simplifications of these types have been made, the resulting expressions have the following properties (to be cited as our agreements): No one of the radicals of highest order n in x_1 is equal to a rational function, with rational coefficients, of the remaining radicals of order n and the radicals of lower orders, while no one of the radicals of order $n-1$ is equal to a rational function of the remaining radicals of order $n-1$ and the radicals of lower orders, etc.

Let $\sqrt{k}$ be a radical of highest order n in x_1. Then

$$x_1=\frac{a+b\sqrt{k}}{c+d\sqrt{k}},$$

where a, b, c, d do not involve $\sqrt{k}$, but may involve other radicals. If $d=0$, then $c\neq 0$ and we write e for a/c, f for b/c, and get

$$\tag{6} x_1 =$$

* See Problem 9, § 105. But we do not actually use here this geometrical interpretation of (5). Any similar compound radical would serve as an illustration.

where neither e nor f involves $\sqrt{k}$. If $d\neq 0$, we derive (6) by multiplying the numerator and denominator of the fraction for x_1 by $c-d\sqrt{k}$, which is not zero since $\sqrt{k}=c/d$ would contradict our above agreements.

By hypothesis, x_1 in (6) is a root of equation (4). After expanding the powers and replacing the square of $\sqrt{k}$ by k, we see that

$$(7) \qquad (e+f\sqrt{k})^3+p(e+f\sqrt{k})^2+q(e+f\sqrt{k})+r=A+B\sqrt{k},$$

where A and B are certain polynomials in e, f, k and the rational numbers p, q, r. Thus $A+B\sqrt{k}=0$. If $B\neq 0$, $\sqrt{k}=-A/B$ is a rational function, with rational coefficients, of the radicals, other than $\sqrt{k}$, in x_1, contrary to our agreements. Hence $B=0$ and therefore $A=0$.

When $e-f\sqrt{k}$ is substituted for x in the cubic function (4), the result is the left member of (7) with $\sqrt{k}$ replaced by $-\sqrt{k}$, and hence the result is $A-B\sqrt{k}$. But $A=B=0$. This shows that

$$(8) \qquad x_2=e-f\sqrt{k}$$

is a new root of our cubic equation. Since the sum of the three roots is equal to $-p$ by § 16, the third root is

$$(9) \qquad {}_3=-p-x_1-x_2=-p-2e.$$

Now p is rational. If also e is rational, x_3 is a rational root and we have reached our goal. We next make the assumption that e is irrational and show that it leads to a contradiction. Since e is a component part of the constructible root (6), its only irrationalities are square roots. Let $\sqrt{s}$ be one of the radicals of highest order in e. By the argument which led to (6), we may write $e=e'+f'\sqrt{s}$, whence, by (9),

$$(10) \qquad x_3=g+h\sqrt{s}, \qquad (h\neq 0)$$

where neither g nor h involves $\sqrt{s}$. Then by the argument which led to (8), $g-h\sqrt{s}$ is a root, different from x_3, of our cubic equation, and hence is equal to x_1 or x_2 since there are only three roots (§ 13). Thus

$$g-h\sqrt{s}=e\pm f\sqrt{k}.$$

By definition, $\sqrt{s}$ is one of the radicals occurring in e. Also, by (10), every radical occurring in g or h occurs in x_3 and hence in $e=\frac{1}{2}(-p-x_3)$, by (9), p being rational. Hence $\sqrt{k}$ is expressible rationally in terms

of the remaining radicals occurring in e and f, and hence in x_1, whose value is given by (6). But this contradicts one of our agreements.

28. Regular Polygon of Nine Sides. In such a polygon the angle subtended at the center by one side is $\frac{1}{9}\cdot 360° = 40°$. But this angle cannot be constructed by ruler and compasses since angle 120° cannot be so trisected (end of § 26). Hence *it is impossible to construct by ruler and compasses a regular polygon of nine sides.*

29. Regular Polygon of Seven Sides. The angle B subtended at the center by one side of such a polygon contains 360/7 degrees. As in § 24, if we could construct B with ruler and compasses, we could so construct a line of length $x = 2 \cos B$. We have

$$\cos 4B = \cos (360° - 4B) = \cos (7B - 4B) = \cos 3B,$$

$$\cos 4B = 2 \cos^2 2B - 1 = 2\,(2 \cos^2 B - 1)^2 - 1,$$

$$\cos 3B = 4 \cos^3 B - 3 \cos B.$$

Hence

$$2(2 \cos^2 B - 1)^2 - 1 = 4 \cos^3 B - 3 \cos B.$$

Multiply all terms by 2 and replace $2 \cos B$ by x. We get

$$(x^2-2)^2 - 2 = x^3 - 3x, \qquad x^4 - 4x^2 - (x^3 - 3x - 2) = 0,$$

$$x^2(x^2-4) - (x-2)(x^2+2x+1) = 0, \qquad (x-2)(x^3+x^2-2x-1) = 0.$$

But if $x = 2$, then $\cos B = 1$, whereas B is an acute angle. Hence

$$x^3 + x^2 - 2x - 1 = 0. \tag{11}$$

Any rational root must be an integer. But neither divisor ± 1 of the constant term is a root. Thus (11) has no rational root. Theorem 1 shows that *it is impossible to construct with ruler and compasses a regular polygon of seven sides.*

The question as to which regular polygons can be constructed and which can not is answered in Chapter XII.

30. Angles Which Can Be Trisected and Those Which Can Not. If two integers p and q have no common divisor >1, they are called *relatively prime*, and p is said to be *prime to q*. Then if p divides qm, p must divide m.

For example, 6 is not prime to 15, but 6 is prime to 35. Then, if 6 divides $35m$, 6 must divide m.

THEOREM 2. *If* p *and* q *are relatively prime integers and* $q>1$, *it is impossible to trisect with ruler and compasses an angle* A *whose cosine is* p/q *in any of the following three cases:*

(i) q *is not divisible by an integral cube* >1;

(ii) $q=c^3d$, $c>1$, $d>2$, d *is not divisible by a cube* >1;

(iii) $q=c^3d$, $c>1$, $d=1$ *or* 2, *if there is no integral root* r *numerically* $<2c$ *of*

$$r^3-3rc^2=2p/d. \tag{12}$$

But if $q=c^3d$, $c>1$, $d=1$ *or* 2, *and if there is an integral root* r *numerically* $<2c$ *of* (12), *then angle* A *can be trisected with ruler and compasses.*

The four cases exhaust all possibilities apart from the trivial case in which A is a multiple of 180°, whence $\cos A=\pm 1$, $q=1$. In fact, if (i) does not hold, q is divisible by a cube >1 and we define c^3 as the largest cube which divides q and define d as q/c^3, so that d is not divisible by a cube >1. Our theorem, therefore, decides whether or not we can trisect an angle whose cosine is any given rational number.

Proof. In cases (i), (ii), (iii), it suffices in view of Lemma 1 and Theorem 1 to prove that there is no rational root of equation (1), which is now

$$x^3-3x-2p/q=0. \tag{13}$$

Suppose that (13) has the root $x=r/s$, where r and s are relatively prime integers and $s>0$. By the substitution of r/s for x and clearing of fractions, we get

$$qt=2ps^3, \qquad t=r^3-3rs^2. \tag{14}$$

If a prime number divided both t and s^3, it would divide s, r^3, and r, and would be a common factor >1 of r and s, contrary to hypothesis. This proves that s^3 is prime to t. Hence by the first equation (14), s^3 divides q.

In case (i), we conclude that $s^3=1$, so that $s=1$. Then the first equation (14) shows that q divides $2p$. But q is prime to p. Hence q divides 2. But $q>1$ by hypothesis. Hence $q=2$. Since $p/q=p/2$ is the value of $\cos A$, p is numerically $\leqq 2$. If $p=\pm 2$, p would have the factor 2 in common with $q=2$. Hence $p=\pm 1$. Then (13) becomes $x^3-3x\mp 1=0$. For the upper sign, this is equation (2), which was shown to have no

rational root. For the lower sign, we replace x by $-x$ and again get (2). Thus the assumption that equation (13) has a rational root r/s has led to a contradiction. This proves Theorem 2 for case (i).

Excluding case (i), we have $q=c^3d$, where $c>1$ and d is not divisible by a cube >1. We saw that s^3 divides $q=c^3d$. Let G denote the greatest common divisor of s and c. Thus $s=GS$, $c=GC$, where S and C are relatively prime integers. Then $s^3=G^3S^3$ divides $q=G^3C^3d$. Hence S^3 divides C^3d, but is prime to C^3. Hence S^3 divides d. Thus $S^3=1$, $S=1$. Hence $s=G$, $c=sC$. The first equation (14) now becomes $s^3C^3dt=2ps^3$, or $C^3dt=2p$. Thus C^3 divides $2p$. But C divides $sC=c$, so that C^3 divides c^3 and hence divides $q=c^3d$. Since q is prime to p, the divisor C^3 of q is prime to p. But C^3 divides $2p$. Hence C^3 divides 2, so that $C=1$. Thus

$$s=c, \qquad dt=2p, \tag{15}$$

so that d divides $2p$. But d is a factor of q and hence is prime to p. Thus d divides 2 and is a positive integer. Hence $d=1$ or 2. This proves Theorem 2 for case (ii).

By (15) and the second equation (14), we have (12). Thus

$$\frac{p}{q}=\frac{r^3-3rc^2}{2c^3}=\frac{4r^3-3rb^2}{b^3}, \qquad b=2c. \tag{16}$$

Suppose that the numerical value of r is $\geqq b$. If $r\geqq 0$, then $r\geqq b$ and

$$4r^3-3rb^2\geqq 4rb^2-3rb^2=rb^2\geqq b^3, \qquad p\geqq q.$$

But if $r=-R$, where $R\geqq b$, the negative of $4r^3-3rb^2$ is $\geqq b^3$ by a like proof, so that $-p\geqq q$. But p/q is the value of $\cos A$ and hence is numerically $\leqq 1$. If $\pm p=q$, p and q would not be relatively prime unless $q=1$, contrary to hypothesis. Hence $\pm p<q$. This contradiction shows that our supposition is false, whence r is numerically $<b=2c$. This proves Theorem 2 for the case (iii).

Finally, we make the assumptions in the last sentence of Theorem 2. We saw that (12) implies (16), which therefore holds for an integer r numerically $<b$. The last statement in Theorem 2 now follows from

LEMMA 4. *Angle* A *can be trisected with ruler and compasses if*

$$\cos A=\frac{4r^3-3rb^2}{b^3},$$

where the integer r *is numerically less than the positive integer* b.

Proof. Let a denote the numerical value of r. By use of parallel lines we can construct (as in Fig. 7) a line whose length is a/b, when the unit of length is given. As in Fig. 5 or Fig. 6, we can construct an angle B such that $\cos B = r/b$. By trigonometry,

$$\cos 3B = 4\cos^3 B - 3\cos B = \frac{4r^3}{\quad}$$

Hence $\cos 3B = \cos A$, so that $A = n \cdot 360° \pm 3B$, where n is an integer. Since we can construct the exterior angle 120° of an equilateral triangle, we can construct angle $\frac{1}{3}A = n \cdot 120° \pm B$.

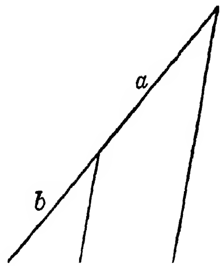


FIG. 7

To tabulate rational values p/q of $\cos A$ such that angle A can be trisected, we assign arbitrary integral values to c and r, where $c > 1$ and r is numerically $< 2c$; then take $q = c^3 d$, $d = 1$ or 2, and determine p by (12). We shall find the cases in which p is an integer.

We may discard any case in which r and c have a common factor $f > 1$. In fact, if $r = fR$, $c = fC$, then $p = f^3 P$, $q = f^3 Q$, and the simplified form P/Q of p/q is obtained from (12) and $q = c^3 d$ written in capital letters.

Let $d = 1$. If c is even, write $c = 2C$. Then (12) becomes $r^3 - 12rC^2 = 2p$. Thus r^3 is even, so that $r = 2R$. Hence $p = 4P$, $P = R^3 - 3RC^2$, $q = 4Q$, $Q = 2C^3$. But this simplified form P/Q of p/q is obtained from the case $d = 2$ of (12) and $q = c^3 d$ written in capital letters. Hence we may assume that c is odd. See Problem 2 below.

Let $d = 2$. If $r = 2R$, write $p = 2P$, $q = 2Q$. Then $Q = c^3$, $P = 4R^3 - 3Rc^2$. The simplified form P/Q of p/q is obtained from the case $d = 1$ of (12), and $q = c^3$, with p and q replaced by P and Q and with $r = 2R$. Hence we may assume that r is odd. Then if also c were odd, $p = r^3 - 3rc^2$ would be even, $p = 2p_1$, $q = 2q_1$, $q_1 = c^3$, and p_1 and q_1 would be obtained from the case $d = 1$ of (12) and $q = c^3$ with the same r and c, but with p and q replaced by p_1 and q_1. Hence we may assume that c is even. See Problem 1.

If we change the sign of r, we merely change the sign of p in (12). Hence we need only make computations with positive values of r.

PROBLEMS

Carry out the preceding tabulation with $c \leqq 4$ if $d = 2$, and $c \leqq 5$ if $d = 1$, and hence prove that we can trisect with ruler and compasses an angle whose cosine is

1. $\pm 11/16$, $\pm 9/16$, $\pm p/128$ for $p = 7$, 47, 115 or 117.

2. ± 1, $\pm p/27$ for $p = 5$, 13, 22 or 23; $\pm P/125$ for $P = 27$, 37, 44, 71, 91, 99, 117, 118.

Prove that it is impossible, with ruler and compasses,

3. To trisect an angle whose cosine is an irreducible fraction numerically $\leqq 1$ whose denominator is $< 343 = 7^3$, except for the 38 fractions in Problems 1 and 2. There are

about 71100 of the former fractions. Hence about one in 2000 of such angles can be trisected.

4. To construct a regular polygon of nm sides if one of n sides can not be constructed. Hint: If $m=2$, join alternate vertices of the former.

5. To construct regular polygons of 14, 21, 18, or 36 sides.

6. To divide angle 100° or 200° into five equal parts.

7. To construct the edge x of a cube whose volume is double that of a given cube (whose edge is taken as the unit of length). This is the ancient Greek problem of the duplication of a cube.

8. We can construct one-fourth or one-eighth of any angle A, but not one-fifth of A if $A=\frac{5}{3}B$, where B is any angle which can not be trisected.

For further problems and related results, see § 109.

31. Trisection with Other Tools. Heretofore we have allowed only the drawing of circles and straight lines.

Archimedes allowed the use of compasses and a graduated ruler. To so trisect angle CAB, draw the semicircle BCD (Fig. 8). Rotate the graduated ruler about C until it shows a segment GF whose length, measured by the ruler, is equal to the measured length of AB. In the isosceles triangle AGF, angle GAF is equal to angle F. Its exterior angle AGC is therefore $2F$. In the isosceles triangle ACG, angle ACG is therefore equal to $2F$. The exterior angle CAB of triangle ACF is the sum of angles ACG and F and hence is $3F$. Thus F is one-third of angle CAB.

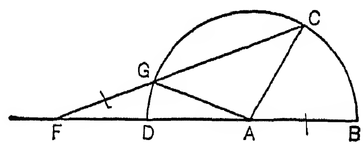


FIG. 8

But we may dispense with the graduated ruler and use merely a ruler. First, place an edge of the ruler along AB so that one end E is at A, and mark on the edge the point P which coincides with B. Second, place the ruler so that the end E is on the diameter DB produced and so that its edge contains point C (Fig. 9). Third, move the ruler so that E slides along the diameter, with C always on the edge, until the point P takes a position on the semicircle. At that moment, P will coincide with G and E with F in Fig. 8.

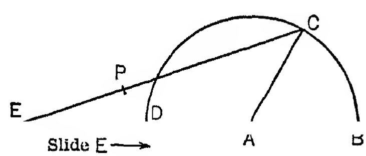


FIG. 9

Although the points F and G were located (and hence angle CAB was trisected) by a mechanical use of the ruler and compasses as the only tools,

those points were not *constructed* geometrically by the drawing of lines and circles. Proofs involving the movements of the ruler are debarred from elementary geometry.

An example of such a debarred proof is the following argument that the sum of the angles of a plane triangle ABC is 180°. Place the ruler along the base AB. Then rotate it about point B until the ruler lies along side BC. Then rotate it about point C until it lies along side CA. Finally rotate it about point A until it again lies along the base AB. The total amount of rotation is evidently 180°.

If we apply the same argument to a spherical triangle ABC whose sides are arcs of great circles (cut out of the surface of the sphere by planes through its center O) and if we use such an arc as ruler, we see that the ruler in its final position along the base has its initial direction reversed, i.e., turned through 180°. But* the three separate angles of rotation of the ruler are equal to the angles of the spherical triangle, and the sum of the latter angles is known to exceed 180°.

The early Greeks gave many methods to trisect any angle A by employing various curves as tools. For example, grant the use of the "cubic parabola" which is the graph of $y=x^3$ (Fig. 13, § 44). The abscissas x of its points of intersection with the line $y=3x+c$, where $c=2\cos A$, are evidently the real roots of equation (1). One of its roots is $2\cos\frac{1}{3}A$ and gives the required angle $\frac{1}{3}A$.

To give another example, we may use the points of intersection of the parabola $y=x^2$ and the circle through the origin having the center $(\frac{1}{2}c, 2)$.

* The explanation of the apparent paradox is simple. The effect of applying the rotation through angle b about the radius OB as axis and then applying the rotation through angle c about the radius OC is usually not the same as applying a rotation through angle $b+c$. Consider the classic example of three mutually perpendicular axes of coordinates in space. Let X, Y, Z denote the rotations through 180° about the x-axis, y-axis, z-axis, respectively. Rotation Z carries any point (x, y) in the xy-plane to the point $(-x, -y)$. Rotation X carries the latter to $(-x, y)$. But rotation Y itself carries (x, y) to $(-x, y)$. Hence the effect of applying rotations Z and X in succession is the same as applying rotation Y.

CHAPTER V

Solution by Radicals of Cubic and Quartic Equations

32. Introductory Remarks. Methods of finding an approximation to a root in terms of decimals (as 2.0945+) are discussed in the later Chapter VIII. Here we demand exact expressions for the roots in terms of radicals, such as square roots and cube roots. The following is one of the most important results in mathematics. While the general (literal) equation of degree 2, 3, or 4 is solvable by radicals, that of degree 5 or higher is not solvable in terms of radicals. We shall prove the first part. But the second part is beyond the scope of this book, since it requires the theory of groups.

If we cube $3-\sqrt{2}$ we get $45-29\sqrt{2}$, so that the real cube root of the latter is $3-\sqrt{2}$. But it is rarely possible to express $\sqrt[3]{a+b\sqrt{2}}$ in the simpler form $u+v\sqrt{2}$, where a, b, u, v are all rational numbers; similarly when $\sqrt{2}$ is replaced by $\sqrt{3}$ or $\sqrt{5}$, etc. But if answers are given to a problem on the solution of a cubic equation by radicals and if one answer is a rational root r, the student feels obliged to attempt to simplify the cube roots by guessing the values of u and v and testing each guess by cubing $u+v\sqrt{2}$. Such a problem is therefore not a reasonable one. Solution by radicals could be abandoned and the following earlier method used. First we find r by the method for rational roots; second, we divide out the factor $x-r$; third, we solve the depressed quadratic equation.

To insure that our problems on numerical cubic equations shall all be reasonable and not solvable by the earlier method, we shall propose only equations having no rational root. It will be shown that if there then arises a radical like $\sqrt[3]{2+3\sqrt{5}}$ it cannot be expressed in the form $u+v\sqrt{5}$, with u and v rational. Hence the student should not waste his time attempting such impossible simplifications.

33. Reduced Cubic Equation. If, in the general cubic equation

$$x^3+bx^2+cx+d=0, \tag{1}$$

we set $x=y-b/3$, we obtain the *reduced cubic equation*

$$y^3+py+q=0, \tag{2}$$

lacking the square of the unknown y, where

$$p=c-\frac{b^2}{3}, \qquad -\frac{bc}{3}+\frac{2b^3}{27}. \tag{3}$$

After finding the roots y_1, y_2, y_3 of (2), we shall know the roots of (1):

$$x_1=y_1-\frac{b}{3}, \qquad x_2=y_2-\frac{b}{3}, \qquad x_3=y_3-\frac{b}{3}. \tag{4}$$

34. Algebraic Solution of the Reduced Cubic Equation. We shall employ the method which is essentially the same as that given by Vieta in 1591. We make the substitution

$$y=z-\frac{p}{3z} \tag{5}$$

in (2) and obtain

$$z^3-\frac{p^3}{27z^3}+q=0,$$

since the terms in z cancel, and likewise the terms in $1/z$. Thus

$$z^6+qz^3-\frac{p^3}{27}=0. \tag{6}$$

Solving this as a quadratic equation for z^3, we obtain

$$z^3=-\frac{q}{2} \tag{7}$$

By § 5, any number has three cube roots, two of which are the products of the remaining one by the imaginary cube roots of unity:

$$\omega=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}i, \qquad \omega^2=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}i. \tag{8}$$

We can choose particular cube roots

$$A=\sqrt[3]{-\frac{q}{2}+\sqrt{R}}, \qquad B=\sqrt[3]{-\frac{q}{2}-\sqrt{R}}, \tag{9}$$

such that $AB=-p/3$, since the product of the numbers under the cube root radicals is equal to $(-p/3)^3$. Hence the six values of z are

$$A, \quad \omega A, \quad \omega^2 A, \quad B, \quad \omega B, \quad \omega^2 B.$$

These can be paired so that the product of the two in each pair is $-p/3$:

$$AB=-\frac{p}{3}, \qquad \omega A\cdot\omega^2 B=-\frac{p}{3}, \qquad \omega^2 A\cdot\omega B=-\frac{p}{3}.$$

Hence with any root z is paired a root equal to $-p/(3z)$. By (5), the sum of the two is a value of y. Hence the *three* values of y are

$$(10) \qquad y_1=A+B, \qquad y_2=\omega A+\omega^2 B, \qquad y_3=\omega^2 A+\omega B.$$

It is easy to verify that these numbers are actually roots of (2). For example, since $\omega^3=1$, the cube of y_2 is

$$A^3+B^3+3\omega A^2B+3\omega^2AB^2=-q-p(\omega A+\omega^2B)$$

by (9) and $AB=-p/3$.

The numbers (10) are known as *Cardan's formulas* for the roots of a reduced cubic equation (2). The expression $A+B$ for a root was first published by Cardan in his *Ars Magna* of 1545, although he had obtained it from Tartaglia under promise of secrecy.

The case in which R is negative is postponed to §§ 37–38. We assume now that R is positive, so that A and B in formulas (9) may be chosen to be the real cube roots.

EXAMPLE 1. Solve $y^3+6y+2=0$.

Solution. Here $p=6$, $q=2$, $R=9$, whence $A=\sqrt[3]{2}$, $B=\sqrt[3]{-4}$. By (10) the desired roots are

$$\sqrt[3]{2}-\sqrt[3]{4}, \quad \omega\sqrt[3]{2}-\omega^2\sqrt[3]{4}, \quad \omega^2\sqrt[3]{2}-\omega\sqrt[3]{4}.$$

Similarly, R is a perfect square in Problems 1–43.

PROBLEMS

Find all of the roots of the following equations:

1. $y^3+9y-6=0$, *Ans.* $\sqrt[3]{9}-\sqrt[3]{3}$, $\omega\sqrt[3]{9}-\omega^2\sqrt[3]{3}$, $\omega^2\sqrt[3]{9}-\omega\sqrt[3]{3}$.
2. $y^3-9y-12=0$, *Ans.* as in Problem 1 with all signs $+$.
3. $y^3-6y-6=0$.
4. $y^3+18y+6=0$.
5. $y^3+15y-20=0$.
6. $y^3-15y-30=0$.
7. $y^3+21y-42=0$.
8. $y^3+12y+12=0$.
9. $y^3-12y-20=0$.
10. $y^3-18y-30=0$, *Ans.* $A^3=18$, $B^3=12$

11. $y^3+18y-30=0$. **12.** $y^3+30y+30=0$, *Ans.* $A^3=20$, $B^3=-50$.

13. $x^3-9x^2-9x-15=0$, *Ans.* $3+\sqrt[3]{24}+\sqrt[3]{72}$, 3- etc.

14. $x^3-6x^2-4=0$. **15.** $y^3+36y+12=0$, *Ans.* $A^3=36$,

16. $x^3-3x^2-18x-36=0$. **17.** $y^3+18y+15=0$, *Ans.* $A=\sqrt[3]{9}$, $B=-2\sqrt[3]{3}$.

18. $x^3-6x^2-12x-8=0$. **19.** $y^3+42y+7=0$,

20. $x^3-6x^2-6x-14=0$. **21.** $y^3+12y-30=0$, $A=2\sqrt[3]{4}$, $B=-$

22. $y^3-12y-34=0$. **23.** $y^3+18y+50=0$, $A=\sqrt[3]{4}$, $B=-$

24. $y^3+30y+15=0$. **25.** $y^3-18y-110=0$, $A=3\sqrt[3]{4}$, $B=\sqrt[3]{\ }$

26. $y^3-18y-75=0$. **27.** $y^3+18y-69=0$. **28.** $y^3-18y-33=0$.

29. $y^3-30y-65=0$. **30.** $y^3+54y-9=0$. **31.** $y^3+66y-33=0$.

32. $y^3-18y-58=0$. **33.** $y^3+36y+92=0$. **34.** $y^3+45y+30=0$.

35. $y^3-21y-56=0$. **36.** $y^3+60y+20=0$. **37.** $y^3+78y-65=0$.

38. $y^3-18y-42=0$. **39.** $y^3+42y+70=0$. **40.** $y^3-30y-70=0$.

41. $y^3-3kty-t(t+k^3)=0$. **42.** $y^3-3sty-st(s+t)=0$.

43. $y^3+3wy+2vw=0$, $w=u^2-v^2$, u and v arbitrary.

EXAMPLE 2. Solve $y^3+3y+2=0$.

Solution. Comparison with equation (2) gives $p=3$, $q=2$. Then $R=2$ by (7). Hence formulas (9) give

Substitution of these values into (10) gives the desired roots.

To test the proposed equation for integral roots, note that there is evidently no positive root, while neither of the negative divisors -1 and -2 of the constant term is a root, by trial. Then, by Theorem 6 of § 21, there is no rational root. Suppose that A could be expressed in the simpler form $u+v\sqrt{2}$, where u and v are rational numbers. Then * would $B=u-v\sqrt{2}$, so that our equation would have the root $A+B=2u$, which is rational. This contradiction shows that we cannot express a cube root A of $-1+\sqrt{2}$ in the form $u+v\sqrt{2}$. Similarly, we cannot simplify B.

PROBLEMS

For the following equations exhibit A and B and prove that these two cube roots cannot be simplified.

1. $y^3+3y+6=0$. Here $A=\sqrt[3]{-3+\sqrt{10}}$, $B=\sqrt[3]{-3-\sqrt{10}}$.

2. $y^3+3y+8=0$. **3.** $y^3+6y+4=0$.

4. $y^3+6y+6=0$. **5.** $y^3+6y+8=0$.

6. $y^3+9y+2=0$. **7.** $y^3+9y+4=0$.

8. $x^3-3x^2+12x-12=0$. **9.** $x^3-6x^2+21x-18=0$.

* The proof is like that in the first part of § 37.

35. Discriminant. For any equation in which the coefficient of the highest power of the unknown is unity, the product of the squares of the differences of its roots is called its *discriminant.* Thus the discriminant of equation (2) is

$$(y_1-y_2)^2(y_1-y_3)^2(y_2-y_3)^2. \tag{11}$$

We shall compute this product by using formulas (10), $\omega^3=1$, and $\omega^2+\omega+1=0$.

$$y_1-y_2=(1-\omega)(A-\omega^2B), \qquad y_1-y_3=(1-\omega^2)(A-\omega B),$$

$$y_2-y_3=(\omega-\omega^2)(A-B),$$

$$(1-\omega)(1-\omega^2)=3, \qquad \omega-\omega^2=\sqrt{3}i.$$

Since 1, ω, ω^2 are the cube roots of unity,

$$(x-1)(x-\omega)(x-\omega^2)\equiv x^3-1,$$

identically in x. Taking $x=A/B$, we see that

$$(A-B)(A-\omega B)(A-\omega^2B)=A^3-B^3=2\sqrt{R},$$

by formulas (9). Hence

$$(y_1-y_2)(y_1-y_3)(y_2-y_3)=6\sqrt{3}\sqrt{R}i.$$

Squaring, and noting that $-108R=-4p^3-27q^2$ by (7), we obtain the following result, which should be memorized.

Theorem 1. *The discriminant of* $y^3+py+q=0$ *is* $-4p^3-27q^2$.

Relations (4) give at once

$$x_1-x_2=y_1-y_2, \qquad x_1-x_3=y_1-y_3, \qquad x_2-x_3=y_2-y_3.$$

Hence by the definition (11) of a discriminant we get

Theorem 2. *The discriminant Δ of the general cubic equation* (1) *is equal to the discriminant of the corresponding reduced cubic equation* (2). *Hence*

$$\Delta=18bcd-4b^3d+b^2c^2-4c^3-27d^2. \tag{12}$$

This expression for Δ was found from Theorem 1 by using the values of p and q given by relations (3).

It is sometimes convenient to employ a cubic equation

$$ax^3+bx^2+cx+d=0 \quad (a\neq 0), \tag{13}$$

in which the coefficient of x^3 has not been made unity by division. The product P of the squares of the differences of its roots is evidently derived from expression (12) by replacing b, c, d, by b/a, c/a, d/a. Hence

$$a^4P \tag{14}$$

This expression (and not P itself) is called the discriminant of equation (13).

36. Number of Real Roots of a Cubic Equation.

THEOREM 3. *A cubic equation with real coefficients has three distinct real roots if its discriminant Δ is positive, a single real root and two conjugate imaginary roots if Δ is negative, and at least two equal real roots if Δ is zero.*

Proof. If the roots x_1, x_2, x_3 are all real and distinct, the square of the difference of any two is positive and hence Δ is positive.

If x_1 and x_2 are conjugate imaginaries and hence x_3 is real (§ 16), then $(x_1-x_2)^2$ is negative. Since x_1-x_3 and x_2-x_3 are conjugate imaginaries, their product is positive. Hence Δ is negative.

If $x_1=x_2$, evidently $\Delta=0$. If x_1 were imaginary, its conjugate would be a second double root by Theorem 10 of Chapter II. This absurdity shows that the equal roots of a real cubic equation are real.

We have now proved the converse of Theorem 3.

Theorem 3 follows from these three results by formal reasoning. For example, if Δ is negative, one root is real and the remaining two are conjugate imaginaries. Otherwise, either the three roots are all real and distinct (and Δ would be positive by our first case, contrary to our hypothesis that Δ is negative), or else two roots are real and equal (and Δ would be zero).

PROBLEMS

Compute the discriminant Δ and find the number of real roots of

1. $y^3-2y-6=0$. **2.** $y^3-48y+64\sqrt{2}=0$, *Ans.* $\Delta=2\cdot 27\cdot 8^4$, three.

3. $y^3-9y+6\sqrt{3}=0$. **4.** $2x^3-6x^2-1=0$, *Ans.* $\Delta=-27\cdot 36$, one.

5. $6x^3+6x^2-1=0$. **6.** $y^3-4y+1=0$, *Ans.* $\Delta=229$, three.

7. In the study of parabolic orbits occurs the equation $\tan \frac{1}{2}v + \frac{1}{3}\tan^3 \frac{1}{2}v = t$. Prove that there is a single real root and that it has the same sign as t.

8. In the problem of three astronomical bodies occurs the equation $x^3 + ax + 2 = 0$. Prove that it has three real roots if and only if $a \leqq -3$.

9. There is a single real point of intersection of the parabola $y = x^2$ and the hyperbola $xy + 8x + 4y + 3 = 0$. Hint: Transpose the terms involving x and square.

37. Irreducible Case. When the roots of a real cubic equation are all real and distinct, we saw that its discriminant Δ is positive; whence $R = -\Delta/108$ is negative. Then Cardan's formulas present the values of the roots in a form involving two cube roots of conjugate imaginaries. If we could extract these cube roots and thus express A and B in the forms $A = u + vi$ and $B = u - vi$ (see the next footnote), the root $A + B$ of the cubic equation would take the desired real form $2u$, and similarly for the remaining two roots in formulas (10).

We shall attempt to extract these cube roots algebraically since we are here interested in exact solutions of our cubic equation and not approximations to its roots. Given two real numbers s and t, where $t \neq 0$, we seek real numbers u and v such that

$$(u + vi)^3 = s + ti.$$

Cubing and replacing i^2 by -1 and i^3 by $-i$, we find that*

$$u^3 - 3uv^2 = s, \qquad 3u^2v - v^3 = t.$$

Thus $v \neq 0$ since $t \neq 0$, and we may employ $w = u/v$. Replacing u by its value vw, we see that our two relations become

$$(w^3 - 3w)v^3 = s, \qquad (3w^2 - 1)v^3 = t.$$

By division we eliminate v^3 and get

$$w^3 - \frac{3s}{t}w^2 - 3w + \frac{s}{t} = 0.$$

To obtain the reduced cubic equation, we take $w = Y + s/t$ (§ 33). We get

$$Y^3 - 3kY - \frac{2sk}{t} = 0, \qquad k = 1 + \frac{s^2}{t^2}.$$

* These imply that $(u - vi)^3 = s - ti$.

This equation becomes equation (2) if we take $p=-3k$, $q=-2sk/t$. By the definition of R in (7), we now have

$$R=-k^3+\frac{s^2k^2}{t^2}=k^2\left(\frac{s^2}{t^2}-k\right)=-k^2.$$

Hence the first formula (9) becomes

$$A=\sqrt[3]{\frac{sk}{t}+ki}=\sqrt[3]{\frac{k}{t}}\cdot\sqrt[3]{s+ti}.$$

While the first factor is the cube root of a real number and so presents no difficulty, the second factor is exactly the cube root which we started out to find. Hence we have failed in our attempt to find it algebraically. Moreover, any different attack on this problem is certain to fail.*

We have now explained the reasons why the present case $\Delta>0$ is called the "irreducible case."

Abandoning the futile attempt to extract a cube root of $s+ti$ exactly (by algebra), we might resort to the approximate cube roots found by trigonometric tables (§ 5), and then compute approximations to the three roots of the cubic equation by use of Cardan's formulas. But this would involve two types of calculations instead of the single one required in the next section.

38. Trigonometric Solution of a Cubic Equation in the Irreducible Case. Let $\Delta>0$, so that $R<0$. By trigonometry,

$$\cos 3A=4\cos^3 A-3\cos A$$

for every angle A. Replacing A by $A+120°$ and $A+240°$ in turn, we get

$$\cos(3A+360°)=\cos 3A=4\cos^3(A+120°)-3\cos(A+120°),$$

$$\cos(3A+720°)=\cos 3A=4\cos^3(A+240°)-3\cos(A+240°).$$

* It is proved in advanced books that if a cubic equation has rational coefficients and has three real roots no one of which is rational, it cannot be solved in terms of real radicals only. This implies that a cube root of a general complex number cannot be expressed in the form $u+vi$, where u and v involve only real radicals. For, if so, Cardan's formulas could be simplified, in the manner explained earlier, so as to express the roots of the cubic equation in terms of real radicals.

These three formulas show that $\cos A$, $\cos(A+120°)$, and $\cos(A+240°)$ are the three roots of the equation $4z^3-3z=\cos 3A$ and hence of

$$z^3-\tfrac{3}{4}z-\tfrac{1}{4}\cos 3A=0.$$

To solve $y^3+py+q=0$, take $y=nz$; we get

$$z^3+\frac{p}{n^2}z+\frac{q}{n^3}=0.$$

This will be identical with the former equation in z if

$$n=\sqrt{\frac{-4}{3}p}, \qquad \cos 3A=-\frac{1}{2}q\left(\frac{-3}{p}\right)^{\frac{3}{2}},$$

as shown by eliminating n from $-\frac{1}{4}\cos 3A=q/n^3$.

Since $R=p^3/27+q^2/4$ is negative by assumption, p is negative and hence n is real; we take it to be the positive square root. Also, the expression obtained for $\cos 3A$ is real and numerically less than unity. Hence we can find angle $3A$ from a table of cosines. We then readily compute

$$\cos A, \qquad \cos(A+120°), \qquad \cos(A+240°),$$

which we proved are the three values of z. Multiplying them by n, we obtain the values $y=nz$ of the roots of the proposed equation $y^3+py+q=0$.

Example. Solve $y^3-7y+7=0$ by trigonometry.

Solution. Here $n=\sqrt{28/3}$, $\qquad \cos 3A=-\sqrt{27/28}$,

$$\log\cos(180°-3A)=\tfrac{1}{2}(\log 27-\log 28)=9.9921029-10,$$

$$180°-3A=10°\,53'\,36'', \quad A=56°\,22'\,8'', \quad \cos(A+120°)=-\cos(60°-A),$$

$\log\cos A=9.7433872$	$\log\cos 3°\,37'\,52''=9.9991272$	$\log\cos(120°-A)=9.6475284$
$\log n=0.4850183$	$\log n=0.4850183$	$\log n=0.4850183$
0.2284055	0.4841455	0.1325467

after subtracting 10. The final numbers are the logarithms of

$$1.692020, \qquad 3.048916, \qquad 1.356897.$$

Changing the sign of the second, we obtain the three roots.

PROBLEMS

1. $y^3-5y-1=0$, *Ans.* -0.201639, 2.330058, -2.128419.
2. $y^3-18y+12=0$.
3. $y^3-4y+1=0$.
4. $x^3+x^2-2x-1=0$, *Ans.* 1.24698, -1.80194, -0.44504.
5. $y^3-9y+9=0$.
6. $x^3+3x^2-3x-4=0$.
7. $x^3+4x^2-7=0$, *Ans.* 1.164248, -1.772866, -3.391382.
8. $x^3+3x^2-1=0$, *Ans.* 0.53209, -0.65270, -2.87939.
9. A right prism (or cuboid) of height h has a square base whose side is b and whose diagonal is therefore $b\sqrt{2}$. If v denotes the volume and d a diagonal of the prism, then $v=hb^2$ and $d^2=h^2+(b\sqrt{2})^2$. Multiply the last equation by h. Hence $h^3-d^2h+2v=0$. Find h when $d=2$, $v=\frac{1}{2}$. *Ans.* $h=1.8608$ or 0.2541.
10. Solve Problem 9 when $d=\sqrt{6}$, $v=\frac{1}{2}$.
11. y^3-3

39. Solution of the Quartic Equation. The general equation of degree four

$$x^4+bx^3+cx^2+dx+e=0, \tag{15}$$

or quartic equation, becomes after transposition of terms

$$x^4+bx^3=-cx^2-dx-e.$$

The left member contains two of the terms of the square of $x^2+\frac{1}{2}bx$. Hence, by completing the square, we get

$$(x^2+\tfrac{1}{2}bx)^2=(\tfrac{1}{4}b^2-c)x^2-dx-e.$$

Adding $(x^2+\frac{1}{2}bx)y+\frac{1}{4}y^2$ to each member, we obtain

$$(x^2+\tfrac{1}{2}bx+\tfrac{1}{2}y)^2=(\tfrac{1}{4}b^2-c+y)x^2+(\tfrac{1}{2}by-d)x+\tfrac{1}{4}y^2-e. \tag{16}$$

The second member is a perfect square of a linear function of x if and only if its discriminant is zero (Problem 1 of § 9):

$$(\tfrac{1}{2}by-d)^2-4(\tfrac{1}{4}b^2-c+y)(\tfrac{1}{4}y^2-e)=0,$$

which may be written in the form

$$y^3-cy^2+(bd-4e)y-b^2e+4ce-d^2=0. \tag{17}$$

Choose any root y of this *resolvent cubic equation* (17). Then the right member of (16) is the square of a linear function, say $mx+n$. Thus

$$x^2+\tfrac{1}{2}bx+\tfrac{1}{2}y=mx+n \quad \text{or} \quad x^2+\tfrac{1}{2}bx+\tfrac{1}{2}y=-mx-n. \tag{18}$$

The roots of these quadratic equations are the four roots of (16) and hence of the equivalent equation (15). This method of solution is due to Ferrari (1522–1565).

EXAMPLE. Solve $x^4-3x^2+6x-2=0$.

Solution. Here $b=0$, $c=-3$, $d=6$, $e=-2$. Hence (17) becomes

It evidently has the root 1. For $y=1$, (16) becomes

$2=0$ or

The roots $1\pm i$ and $-1\pm\sqrt{2}$ of these two quadratic equations give the answers.

PROBLEMS

For each quartic function 1–100 the resolvent cubic equation has a small integral root, but the quartic has no rational root. For selected problems in each set I, II, III, factor the function or solve the corresponding equation.

I. Quartics having two real and two imaginary roots.

1. $x^4+12x-5$, *Ans.*
2. $x^4+32x-60$.
3.
4. $x^4-x^2+2x-1=0$, *Ans.* $\frac{1}{2}(1\pm\sqrt{-3})$, $\frac{1}{2}(-1\pm\sqrt{5})$.
5. x^4-4x^2+8x-4.
6. $x^4-x^2+10x-4$.
7. x^4-1
8. $x^4-12x^2+24x-5$.
9. $x^4-3x^2+10x-6$.
10. $x^4-9x^2+20x+6$.
11. $x^4-11x^2+28x-6$.
12. $x^4-8x^2+24x+7$.
13. $x^4-10x^2+32x-7$.
14. $x^4-2x^2+12x-8$.
15. $x^4-6x^2+12x-8$.
16. $x^4-9x^2+36x-8$.
17. $x^4-7x^2+28x+8$.
18. $x^4-9x^2+12x+10$.
19. $x^4-27x^2+66x-10$.
20. $x^4-7x^2+14x-10$.
21. $x^4-x^2+14x-10$.
22. x^4-1
23. $x^4+5x^2+22x-10$.
24. $x^4-25x^2+54x+10=0$, *Ans.* $3\pm i$, $-$
25. $x^4-26x^2+72x-11$.
26. $x^4-24x^2+60x+11$.
27. x^4-1
28. $x^4-8x^2+16x+12$.
29. $x^4-5x^2+14x-12$.
30. $x^4-8x^2+16x-12$.
31. $x^4-4x^2+56x-13$.
32. $x^4-22x^2+72x+13$.
33. x^4-2
34. $x^4-24x^2+84x-13$.
35. $x^4-7x^2+20x+14$.
36. x^4-1
37. $x^4-6x^2+16x-15$.
38. $x^4-14x^2+32x-15$.
39. x^4-6
40. $x^4-7x^2+18x-18$.
41. $x^4-13x^2+36x-18=0$, *Ans.*
42. $x^4-5x^2+18x-20$.
43. $x^4-3x^2+18x-20=0$, *Ans.* $1\pm 2i$, -1
44. $x^4-4x^2+20x-25$.
45. $x^4-6x^2+20x-24=0$, *Ans.*

II. Quartics having four real roots.

46. x^4-	47. x^4-	48. x^4-1
49. x^4-7x^2+2x+2.	50. x^4-33x^2+6x+2.	51. x^4-3
52. $x^4-15x^2-12x-2$.	53. x^4-19x^2+4x+2.	54. $x^4-37x^2+18x-2$.
55. x^4-39x^2+6x+2.	56. x^4-20x^2+8x+3.	57. $x^4-32x^2+12x+3$.
58. $x^4-34x^2+24x-3$.	59. x^4-9x^2-6x+4.	60. $x^4-21x^2+12x+4$.
61. $x^4-36x^2+24x-4$.	62. $x^4-31x^2+18x+4$.	63. $x^4-33x^2+30x-4$.
64. x^4-10x^2-8x+5.	65. $x^4-23x^2+20x+6$.	66. x^4-2
67. x^4-31x^2+6x+6.	68. $x^4-35x^2+30x-6$.	69. x^4-2
70. x^4-	71. $x^4-28x^2+36x+7$.	72. $x^4-30x^2+48x-7$.
73. x^4-10x^2-4x+8.	74. x^4-22x^2+8x+8.	75. x^4-8
76. $x^4-34x^2+36x-8$.	77. $x^4-11x^2-6x+10=0$,	*Ans.* $1\pm$
78. x^4-	79. $x^4-24x^2+16x+12=0$,	*Ans.* $2\pm\sqrt{6},\ -2\pm\sqrt{10}$.
80. $x^4-29x^2+6x+12$.	81. $x^4-28x^2+24x+12=0$,	*Ans.* $3\pm\sqrt{}$
82. $x^4-32x^2+48x-12$.	83. $x^4-31x^2+54x-14=0$,	*Ans.* $3\pm\sqrt{}$
84. $x^4-27x^2+30x+14$.	85. $x^4-25x^2+12x+18=0$,	*Ans.* $2\pm\sqrt{7},\ -2\pm\sqrt{10}$.

III. Quartics having four imaginary roots.

86.	87. x^4+2x^2-4x+8.	88. x^4+3x^2+6x+
89. $x^4+3x^2+2x+12$.	90. $x^4+4x^2+4x+15$.	91. $x^4+5x^2+2x+20$.
92. $x^4+8x^2+16x+20$.	93. $x^4-5x^2-4x+30$.	94. $x^4+9x^2+14x+30$.
95. $x^4-4x^2-8x+35$.	96. $x^4-3x^2-12x+40=0$,	*Ans.* $2\pm i,\ -2\pm 2i$.
97. $x^4-3x^2+4x+42$.	98. $x^4+10x^2+12x+40=0$,	*Ans.* $1\pm 3i,\ -1\pm\sqrt{-3}$
99. $x^4-2x^2+8x+48$.	100. $x^4+11x^2+10x+50=0$,	*Ans.* $1\pm 3i,\ -1\pm 2i$.

40. Roots of the Resolvent Cubic Equation. Let y_1 be the root y which was employed in § 39. Let x_1 and x_2 be the roots of the first quadratic equation (18), and x_3 and x_4 the roots of the second. Then

If, instead of y_1, another root y_2 or y_3 of the resolvent cubic equation (17) had been employed in § 39, quadratic equations different from (18) would have been obtained, such, however, that their four roots are x_1, x_2, x_3, x_4, paired in a new manner. The following fact therefore seems plausible.

Theorem 4. *The roots of the resolvent cubic equation* (17) *are*

$$y_1=x_1x_2+x_3x_4,\qquad y_2=x_1x_3+x_2x_4,\qquad y_3=x_1x_4+x_2x_3. \tag{19}$$

Proof. By § 16, we have

$$x_1+x_2+x_3+x_4=-b,\qquad x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4=-d,$$

$$x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4=c,$$

From these four relations and (19) we conclude that

Hence (§ 16) y_1, y_2, y_3 are the roots of the cubic equation (17).

41. Discriminant. The discriminant Δ of the quartic equation (15) is defined to be the product of the squares of the differences of its roots:

$$\Delta=(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2(x_3-x_4)^2.$$

The fact that Δ is equal to the discriminant of the resolvent cubic equation (17) follows at once from relations (19), by which

$$x_3), \qquad y_1-y_3=(x_1$$

$$y_2-y_3=(x_1-x_2)(x_3-x_4), \qquad (y_1-y_2)^2(y_1-y_3)^2(y_2-y_3)^2=\Delta.$$

Hence (§ 35) Δ is equal to the discriminant $-4p^3-27q^2$ of the reduced cubic equation $Y^3+pY+q=0$, obtained from (17) by taking $y=Y+\frac{1}{3}c$. Thus

(20) $$p=bd-4e-\tfrac{1}{3}c^2, \qquad q=-b^2e+\tfrac{1}{3}bcd+\tfrac{8}{3}ce-d^2-\tfrac{2}{27}c^3.$$

THEOREM 5. *The discriminant of any quartic equation* (15) *is equal to the discriminant of its resolvent cubic equation and therefore is equal to the discriminant* $-4p^3-27q^2$ *of the corresponding reduced cubic equation* $Y^3+pY+q=0$, *whose coefficients have the values* (20).

PROBLEMS

Compute the discriminant and show that there is a multiple root for

2. x^4+4x^3+

3. If a real quartic equation has either four distinct real roots or two pairs of conjugate imaginary roots, show that its discriminant Δ is positive. Hence prove by formal reasoning that, if $\Delta<0$, there are exactly two real roots and two imaginary roots.

4. Verify by Problem 3 that $x^4-3x^2-10x-6=0$ has just two real roots.

5. Discuss the points of intersection of the parabola $y=x^2$ and the conic $A(x^2-y)+y^2+2Bxy+2Hx+BH=0$. At an intersection,

Replacing x^2 by y, we get

In (20), verify that $p=0$, $q=-16H^2(B^3+H)^2$. Hence the discriminant is negative if $H\neq 0$, $H\neq -B^3$. Why are there then only two real points of intersection?

6. Find the real points of intersection of $y=x^2$ and $ax^2+y^2-xy-x-(a+5)y-6=0$. *Ans.* (3, 9), (−2, 4).

7. Find a necessary and sufficient condition that the quartic equation (15) shall have one root the negative of another root. Hint: $(x_1+x_2)(x_3+x_4)=c-y_1$. Hence substitute c for y in (17).

8. Verify that $\Delta<0$ for certain of the quartics in case I of § 39.

CHAPTER VI

The Graph of an Equation, Derivatives

42. Use of Graphs in the Theory of Equations. To find geometrically the real roots of a real equation $f(x)=0$, we construct a graph of $y=f(x)$ and measure the distances from the origin O to the points of intersection of the graph and the x-axis. Since the equation of the latter is $y=0$, the abscissas x of the points of intersection are the real roots of $f(x)=0$.

Example. Find graphically the real roots of $x^2-6x-3=0$.

Solution. By plane analytics, the graph of $Y = X^2$ is a parabola (Fig. 10) whose vertex is the origin. We desire the graph of

(1) $\quad y=x^2-6x-3$ or

The latter is reduced to $Y=X^2$ by the transformation $X=x-3$, $Y=y+12$, which corresponds to the choice of new axes parallel to the old: the x-axis ($y=0$ or $Y=12$) is parallel to the old X-axis and is 12 units above it; the y-axis is parallel to the old Y-axis and is 3 units to the left of it. Hence the graph of equation (1) is the same parabola referred to the new axes. The distances, approximately, 6.46 and -0.46, from the new origin O to the intersections of the x-axis and the parabola are the desired roots.

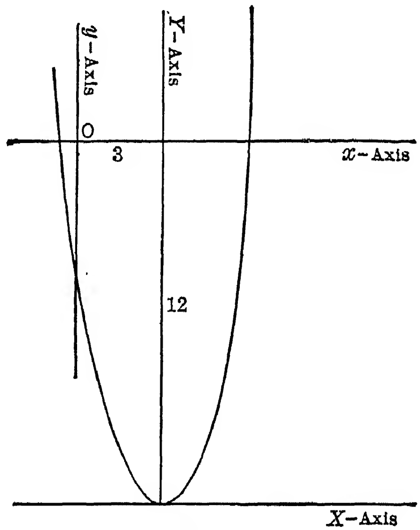


Fig. 10

PROBLEMS

Discuss as to real roots

0. $\quad$ **2.** $x^2-6x+12=0$. $\quad$ **3.** $x^2-6x+9=0$.

4. The real roots of $x^3-px-q=0$ are the abscissas of the intersections of the parabola $y=x^2$ and the circle through the origin having the center $(\frac{1}{2}q, \frac{1}{2}+\frac{1}{2}p)$.

5. In Problem 4, we may replace the circle by the hyperbola $xy-px+q=0$.

43. Caution in Plotting. To find the graph of

(2) $$y=8x^4-14x^3-9x^2+11x-2.$$

we might use successive integral values of x, obtain the points $(-2, 180)$, $(-1, 0)$, $(0, -2)$, $(1, -6)$, $(2, 0)$, $(3, 220)$, all but the first and last of which are shown (by crosses) in Fig. 11, and be tempted to conclude that the graph is a U-shaped curve approximately like that in Fig. 10 and that there are just two real roots, -1 and 2, of

$$8x^4-14x^3-9x^2+11x-2=0. \tag{2'}$$

But both these conclusions would be false. In fact, the graph is a W-shaped curve (Fig. 11) and the additional real roots are $\frac{1}{4}$ and $\frac{1}{2}$.

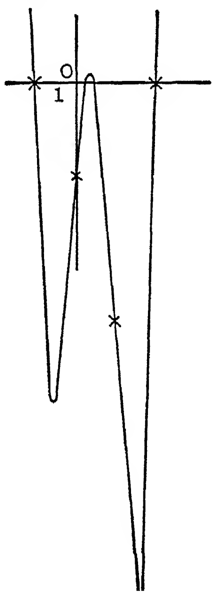


FIG. 11

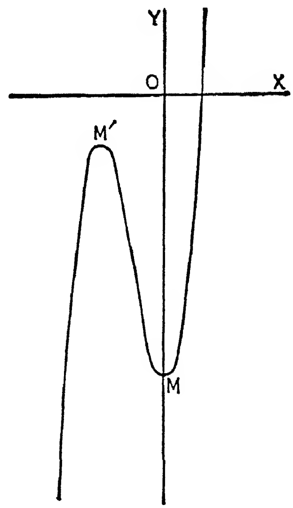


FIG. 12

This example shows that it is often necessary to employ also values of x which are not integers. The purpose of the example was, however, not to point out this obvious fact, but rather to emphasize the chance of serious error in sketching a curve through a number of points, however numerous. The true curve between two points below the x-axis may not cross the x-axis, or may have a peak and actually cross the x-axis twice, or may be an M-shaped or W-shaped curve crossing it four times, etc.

For example, the graph (Fig. 12) of

$$y=x^3+4x^2-11 \tag{3}$$

crosses the x-axis only once; but this fact cannot be established by a graph located by a number of points, however numerous, whose abscissas are chosen at random.

We shall find that correct conclusions regarding the number of real roots may be deduced from a graph constructed with the aid of its bend points, next defined.

44. Bend Points. A point (like M or M' in Fig. 12) is called a *bend point* of the graph of $y=f(x)$ if the tangent to the graph at that point is horizontal and if all the adjacent points of the graph lie below the tangent or all above the tangent. The first, but not the second, condition is satisfied by the point O of the graph of $y=x^3$ given in Fig. 13 (see § 50). In the language of calculus, $f(x)$ has a relative maximum or minimum value at the abscissa of a bend point of the graph of $y=f(x)$.

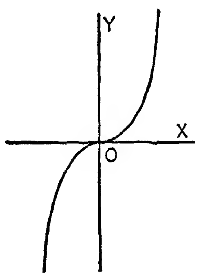


FIG. 13

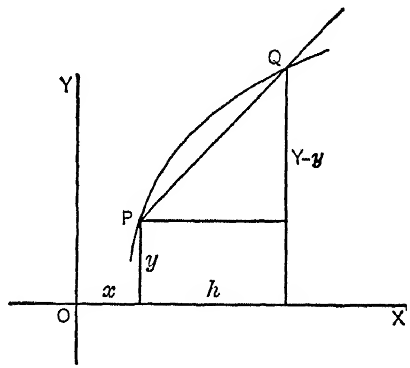


FIG. 14

Let $P=(x,y)$ and $Q=(x+h,\ Y)$ be two points on the graph, sketched in Fig. 14, of $y=f(x)$. By the *slope* of a straight line is meant the tangent of the angle between the line and the x-axis, measured counter-clockwise from the latter. In Fig. 14, the slope of the straight line PQ is

$$\frac{Y-y}{h}=\frac{f(x+h)-f(x)}{h}. \tag{4}$$

For the case of equation (3), we have

$$f(x)=x^3+4x^2-11,$$
$$f(x+h)=(x+h)^3+4(x+h)^2-11.$$

Employing the values of the cube and square, we get

$$f(x+h)=x^3+4x^2-11+(3x^2+8x)h+(3x+4)h^2+h^3. \tag{6}$$

Therefore the slope (4) of the secant PQ of the graph (Figs. 12, 14) of (3) is

Now let the point Q move along the graph toward P. Then h approaches the value zero and the secant PQ approaches the tangent at P. The slope of the tangent at P is therefore the corresponding limit $3x^2+8x$ of the preceding expression. We call $3x^2+8x$ the *derivative* of x^3+4x^2-11.

In particular, if P is a bend point, the slope of the (horizontal) tangent at P is zero, whence $3x^2+8x=0$, $x=0$ or $x=-\frac{8}{3}$. Equation (3) gives the corresponding values of y. The resulting points

are easily shown to be bend points. Indeed, for $x>0$ and for x between -4 and 0, $x^2(x+4)$ is positive, and hence $f(x)>-11$ for such values of x, so that the function (5) has a relative minimum at $x=0$. Similarly, there is a relative maximum at $x=-\frac{8}{3}$. We may also employ the general method of § 52 to show that M and M' are bend points. Since these bend points are both below the x-axis we are now certain that the graph crosses the x-axis only once.

The use of the bend points insures greater accuracy to the graph than the use of dozens of points whose abscissas are taken at random.

45. Derivatives, Taylor's Formula. In formula (6) note that the sum of the terms free of h is $f(x)$. If we add $3x^4$ to the function (5) and so obtain $F(x)=3x^4+f(x)$, we see that $F(x+h)$ is the sum of the second member of relation (6) and $3(x+h)^4=3x^4+\cdots+3h^4$. Thus $F(x+h)$ is the sum of $F(x)$ and terms involving h, h^2, h^3, h^4. In this manner we see without any computation that, if $f(x)$ is any polynomial of degree n,

$$f(x+h)=f(x)+f_1(x)h+\cdots+f_n(x)h^n,$$

in which the polynomials $f_1(x), \cdots, f_n(x)$ have not yet been found. As was done in the special case (6), we could find them by using the binomial theorem (8) for $m=1, \cdots, n$; but this work is laborious and the resulting

expressions for $f_1, \cdots, f_n$ are quite complicated and do not yield their properties as readily as the method which we shall explain.

Note that formula (8) involves denominators $2!, 3!, \cdots$, where $k!$ denotes the product of $1, 2, \cdots, k$ and is read *k factorial*. To take account of these denominators, we shall write $f^{(k)}(x)$ for $k!\,f_k(x)$. Then the preceding formula becomes

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{2!}+\cdots+f^{(n)}(x)\frac{h^n}{n!}, \tag{7}$$

where $f'(x), f''(x), \cdots$ are certain polynomials, as yet unknown, whose properties we seek, rather than explicit expressions for them.

The binomial theorem states that

$$(x+h)^m=x^m+mx^{m-1}h+\frac{m(m-1)}{1\cdot 2}x^{m-2}h^2+\cdots+h^m. \tag{8}$$

Multiply all terms by a constant c. Hence for the special function $f(x)=cx^m$, we see that, in (7),

$$f'(x)=cmx^{m-1}, \qquad f''(x)=cm(m-1)x^{m-2}, \cdots, \tag{9}$$
$$f^{(r)}(x)=cm(m-1)\cdots(m-r+1)x^{m-r}.$$

For the case of the special function (5) we called the coefficient $3x^2+8x$ of h in the expression (6) for $f(x+h)$ the derivative of $f(x)$. In the general case (7), we call $f'(x)$ the (*first*) *derivative* of $f(x)$, and call $f''(x)$ the *second derivative* of $f(x)$, etc. Hence by formulas (9) we see that the first derivative cmx^{m-1} of cx^m is obtained by multiplying the latter by its exponent m and then diminishing its exponent by unity. For example, the derivative of x^3 is $3x^2$, and that of $4x^2$ is $8x$. Also by (9), the second derivative $f''(x)$ of cx^m is seen by the same rule to be equal to the first derivative of $f'(x)$, and in general

$$f^{(r+1)}(x)=cm(m-1)\cdots(m-$$

is seen to be the first derivative of $f^{(r)}(x)$, which is given by the last formula in (9).

These facts, that $f''(x)$ is the derivative of $f'(x)$, that $f'''(x)$ is the derivative of $f''(x)$, etc., hold true not merely for the preceding function cx^m, but also for any polynomial. The latter is the sum of terms cx^m for various values of c and m. Hence it remains only to prove the following fact.

Let $G(x)$, $H(x)$, $\cdots$, $L(x)$ be any polynomials in x. If $s(x)$ denotes their sum, then for their i-th derivatives we shall prove that

$$(10) \qquad s^{(i)}(x)=G^{(i)}(x)+\cdots+L^{(i)}(x) \qquad (i=1, 2, 3, \cdots).$$

Copy equation (7) with f replaced by G; copy (7) with f replaced by H; etc., until (7) has been copied with f replaced by L. Add the members of these copied equations. We get

$$s(x+h)=s(x)+\{G'(x)+\cdots+L'(x)\}h+\cdots$$
$$+\{G^{(i)}(x)+\cdots+L^{(i)}(x)\}\frac{h^i}{i!}+\cdots$$

In (7) we may replace f by s, and get

$$s(x+h)=s(x)+s'(x)h+\cdots+s^{(i)}(x)\frac{h^i}{i!}+\cdots$$

Thus this polynomial in h is equal to the preceding one for all values of h. Hence (§ 14) they are term by term identical. This proves relations (10). We have now completed the proofs of the following important facts.

I. *The derivative of the sum of several polynomials in* x *is equal to the sum of their derivatives.*

II. *The derivative of* cx^m *is* cmx^{m-1}. *In particular, the derivative of the constant* c *is zero.*

III. *If* $f(x)$ *is any polynomial, its second derivative* $f''(x)$ *is the derivative of its first derivative* $f'(x)$, *and in general the derivative of its* r-*th derivative* $f^{(r)}(x)$ *is its* $(r+1)$*th derivative* $f^{(r+1)}(x)$.

By using facts I and II we can find $f'(x)$ by inspection. Then by III we at once find $f''(x)$, etc.

For the cubic function (5), we see that $f'(x)$ is the sum $3x^2+8x$ of the derivatives $3x^2$, $8x$, 0 of its terms x^3, $4x^2$, -11. Again, $f''(x)=6x+8$, $f'''(x)=6$. Thus formula (7) reduces to (6).

With the understanding that f', f'', $\cdots$ are the successive derivatives of $f(x)$, we call (7) *Taylor's formula.*

For later use we shall prove two further facts.

IV. *The derivative of the product* fg *of two polynomials is*

$$fg'+f'g.$$

Proof. Multiply Taylor's formula (7) by the like formula

$$g(x+h)=g(x)+g'(x)h+\tfrac{1}{2}g''(x)h^2+\cdots$$

and note that the coefficient of h in the product is $fg'+f'g$.

V. *The derivative of* $(x-c)^n$ *is* $n(x-c)^{n-1}$ *if* c *is a constant.*

Proof. This is evidently true when $n=1$. Let it be true when $n=m$. Then by IV the derivative of the product $(x-c)(x-c)^m$ is

$$(x-c)m(x-c)^{m-1}+(x-c)^m=(m+1)(x-c)^m.$$

Since V is therefore true when $n=m+1$, it is true for every n by induction.

In view of Taylor's formula (7), the limit of the last fraction in (4) as h approaches zero is $f'(x)$. Hence $f'(x)$ *is the slope of the tangent to the graph of* $y=f(x)$ *at the point* (x, y).

EXAMPLE. Locate the real roots of $f(x)=x^4+x^3-x-2=0$.

Solution. The abscissas of the bend points are the real roots of $f'(x)=4x^3+3x^2-1=0$. We approximate the roots of the latter by means of a graph for $y=f'(x)$; the abscissas of its bend points are the roots 0 and $-\frac{1}{2}$ of $f''(x)=12x^2+6x=0$, so that its bend points are $(0, -1)$ and $(-\frac{1}{2}, -\frac{3}{4})$, whence the graph is of the type shown in Fig. 12. Hence $f'(x)=0$ has a single real root, which is seen to be just less than $\frac{1}{2}$. Thus the single bend point of the graph of $y=f(x)$ is $(\frac{1}{2}, -\frac{37}{16})$, approximately, whence the graph is approximately a U-shaped curve which crosses the x-axis just twice. The two real roots are seen to be $1+$ and $-1\frac{1}{3}$, approximately.

PROBLEMS

1. Find the second derivative of $3x^5+4x^3-7x^2+2$.
2. Find the third derivative of $2x^5-7x^3+x$. *Ans.* $120x^2-42$.
3. Prove V by the binomial theorem when $n=3$ and $n=4$.

Find the bend points of $y=f(x)$ and locate the real roots of

4. $x^3-2x-5=0$, *Ans.* $(.82, -6.09)$, $(-.82, -3.91)$; root $2+$.
5. $x^3-4x+8=0$.
6. $x^3+6x-2=0$; root $\frac{1}{3}-$.
7. $x^3-9x-12=0$.
8. $x^3-18x-30=0$; root $5-$.
9. $x^4-7x^2-20x+14=0$, *Ans.* roots $\frac{1}{2}+$ and $3+$.
10. $x^4-8x^2-24x+7=0$.
11. $x^6-7x^4-3x^2+7=0$, *Ans.* root invervals $(0, 1)$, $(-1, 0)$, $(2.5, 3)$, $(-3, -2.5)$.

46. Continuous and Discontinuous Functions. In case $f(x)$ is a polynomial with real coefficients, we have hitherto located certain points

of the graph of $y=f(x)$ and taken the liberty to join them by an unbroke (continuous) curve. That this is permissible will follow from our nex theorem.

A small change in x causes a small change in x^3. For example,

$$1.99^3=7.8806, \quad 2^3=8, \quad 2.01^3=8.1206, \quad 2.02^3=8.2424.$$

To give precision to the word "small," we shall examine the difference $D=(a+h)^3-a^3$ as follows.

DEFINITION. Let a be a real constant. A real function $f(x)$, including the case of a polynomial with real coefficients, is called *continuous at* $x=a$ if for an arbitrary positive number p the difference

is numerically less than p for all real values of h sufficiently small numerically. In the contrary case, $f(x)$ is called *discontinuous at* $x=a$.

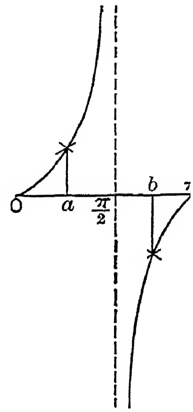


FIG. 15

For example, if x is measured in radians (π radians are equal to 180°, where $\pi=3.1416$, approximately), the trigonometric function $\tan x$ is discontinuous at $x=\frac{1}{2}\pi$. The graph of $y=\tan x$ for $0\leqq x\leqq\pi$ is a broken curve (Fig. 15) consisting of two parts.

Next, when x is real and $\geqq 0$, let $[x]$ denote the largest integer which is $\leqq x$. For example, $[5\frac{1}{3}]=5$, $[5]=5$. Then the graph of $y=[x]$ is composed of infinitely many parallel segments of straight lines each of length unity (Fig. 16). Evidently the function $[x]$ is discontinuous at $x=1$, $x=2$, $x=3$, etc.

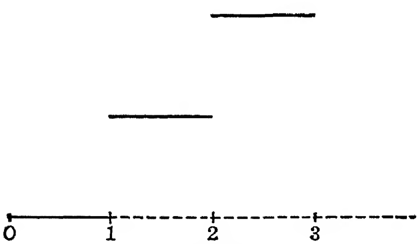


FIG. 16

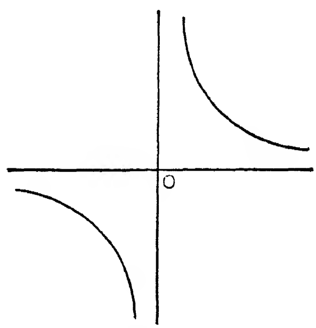


FIG. 17

Again, the function $1/x$ is discontinuous at $x=0$. The reader will recall that the graph (Fig. 17) of $y=1/x$ (or $xy=1$) is an hyperbola whose branches lie in the first and third quadrants.

THEOREM 1. *If* a *is any real constant, any polynomial* f(x) *with real coefficients is continuous at* x=a.

Proof. Taylor's formula (7) with x replaced by a gives

$$D=f'(a)h+\frac{f''(a)}{1\cdot 2}h^2+\cdots+\frac{f^{(n)}(a)}{n!}h^n.$$

Our theorem therefore follows from the next one.

THEOREM 2. *If its coefficients are all real, the function*

$$F=c_1h+c_2h^2+\cdots+c_nh^n \tag{11}$$

is numerically less than any assigned positive number p *for all real values of* h *sufficiently small numerically.*

Proof. Let g denote the greatest of the numerical values of $c_1, \cdots, c_n$. If h is numerically less than k, where $0<k<1$, we see that F is numerically less than

$$g(k+k^2+\cdots+k^n)$$

$$=g\frac{k(1-k^n)}{1-k}<g\frac{k}{1-k}<p, \quad \text{if } k<\frac{p}{p+g}.$$

47. Root between a and b if $f(a)$ and $f(b)$ Have Opposite Signs.

THEOREM 3. *If the coefficients of a polynomial* f(x) *are real and if* a *and* b *are real numbers such that* f(a) *and* f(b) *have opposite signs, the equation* f(x)=0 *has at least one real root between* a *and* b; *in fact, an odd number of such roots, if an* m*-fold root is counted* m *times.*

The only argument* given here is one based upon geometrical intuition. We are stating that, if the points

$$(a, f(a)), \qquad (b, f(b))$$

*An arithmetical proof based upon a refined theory of irrational numbers is given in Weber's *Lehrbuch der Algebra*, ed. 2, vol. 1, p. 123; or any text on analysis.

lie on opposite sides of the x-axis, the graph of $y=f(x)$ crosses the x-axis once, or an odd number of times, between a and b. Indeed, the part of the graph between the vertical lines through the two points is a continuous curve having one and only one point on each intermediate vertical line, since the function has a single value for each value of x.

It is instructive to consider examples of functions $f(x)$ which are not polynomials such that Theorem 3 fails.

First, let $f(x)=\tan x$ and let x be measured in radians. Let $0<a<\frac{1}{2}\pi<b<\pi$. Although $f(a)>0$, $f(b)<0$, Fig. 15 shows that there is no root between a and b of $\tan x=0$.

Second, let $f(x)=1/x$, $a<0$, $b>0$. Although $f(a)<0$, $f(b)>0$, Fig. 17 shows that there is no root between a and b of $f(x)=0$.

Third, let the values of $f(x)$ be those of both $\sqrt{x}$ and $-\sqrt{x}$. The graph of $y^2=x$ is a parabola whose axis is the x-axis. Its points $(4, -\sqrt{4})$ and $(9, \sqrt{9})$ lie on opposite sides of the x-axis; but the parabola does not cross the x-axis between 4 and 9 (the origin is the only point of intersection).

48. Sign of a Polynomial.

THEOREM 4. *When* x *is sufficiently large numerically, any real polynomial*

$$f(x)=c_0x^n+c_1x^{n-1}+\cdots+c_n \qquad (c_0\neq 0) \tag{12}$$

has the same sign as c_0x^n.

We first employ large positive numbers x. We have

$$f(x)\equiv x^n(c_0+F), \qquad F=c_1\left(\frac{1}{x}\right)+c_2\left(\frac{1}{x}\right)^2+\cdots+c_n\left(\frac{1}{x}\right)^n.$$

Apply the result proved for polynomial (11) with h replaced by $1/x$ and with p replaced by the numerical value of c_0. Hence the numerical value of F is less than that of c_0 when $1/x$ is positive and less than a sufficiently small positive number k. Write P for $1/k$. Hence if P is positive and sufficiently large, and if $x>P$, then the numerical value of F is less than that of c_0. Thus c_0+F has the same sign as c_0. Now x is positive, so that $f(x)\equiv x^n(c_0+F)$ has the same sign as c_0x^n.

Second, let $x=-X$, where X is positive. By the first case, $f(-X)$ has the same sign as its first term $(-1)^nc_0X^n$ when X is a sufficiently large positive number. In the last statement replace X by $-x$. Then $f(x)$ has

the same sign as c_0x^n when x is negative and $-x$ is sufficiently large. This completes the proof of the theorem.

The last two conditions on x will be meant when we use the symbol $x=-\infty$. Similarly, when x is positive and sufficiently large, we shall write $x=\infty$. Hence we have

THEOREM 5. *For* $x=\infty$, $f(x)$ *in* (12) *has the same sign as* c_0. *For* $x=-\infty$, $f(x)$ *has the same sign as* c_0 *when* n *is even, but* $f(x)$ *and* c_0 *have opposite signs when* n *is odd.*

Theorem 5 gives useful information about the graph of $y=f(x)$.

I. n even, c_0 positive. The points of the graph with x numerically large are above the x-axis.

II. n odd, c_0 positive. The points of the graph with x large are above the x-axis; those with x negative, but numerically large, are below it.

Case I is illustrated by Figs. 10 and 11. Case II is illustrated by Figs. 12 and 13. Since we may change the signs of all terms of an equation, we shall rarely need a graph when c_0 is negative. Then in I and II we interchange the words above and below.

EXAMPLE. If n is odd, $a>0$, and $l\neq 0$, then the real equation $f(x)=ax^n+\cdots+l=0$ has a real root whose sign is opposite to the sign of l.

Solution. By Theorem 5, $f(\infty)$ is positive, while $f(-\infty)$ is negative. If $l=f(0)$ is negative, Theorem 3 shows that there exists a real root between 0 and ∞; the sign of this positive root is opposite to the negative l. Next, if l is positive, there is a real root between $-\infty$ and 0, and this negative root and l have opposite signs.

PROBLEMS

1. Prove that $8x^3-4x^2-18x+9=0$ has a root between 0 and 1, one between 1 and 2, and one between -2 and -1.

2. Show that $x^4-12x^2-12x-3=0$ has a root between 3 and 4 and another between -3 and -2.

Locate two real roots of

3. $x^4-3x^2+10x-6=0$.

4. $x^4-8x^2-16x+12=0$.

5. $x^4-5x^2+60x-26=0$.

6. $x^4-6x^2-64x-39=0$, *Ans.* 4.6+, $-.6$.

7. Prove that $x^3+ax^2+bx-4=0$ has a positive root.

8. Show that $x^3+ax^2+bx+4=0$ has a negative root.

9. Prove that $x^4+ax^2+bx^2+cx-4=0$ has a positive root and a negative root.

10. Show that any real equation of even degree has a positive root and a negative root if the coefficient of x^n and the constant term have opposite signs.

11. If $a<b<c<d$, and g, h, j, k are positive,

$$x-a \quad x-b \quad x-c \quad x-d$$

has a root between a and b, one between b and c, and one between c and d. If there is a root $>d$. If $t>0$, there is a root $<a$.

49. Multiple Roots. Let r be a root of $f(x)=0$. By the factor theorem, $f(x)$ is divisible by $x-r$. If $f(x)$ is divisible by $(x-r)^m$, but not by $(x-r)^{m+1}$, we call r a root of multiplicity m of $f(x)=0$ (§ 15). We may then write

$$(13) \qquad f(x)=(x-r)^m\, Q(x), \qquad Q(r)\neq 0.$$

Applying the rules IV and V for derivatives, we get

$$(14) \qquad f'(x)=m(x-r)^{m-1}\, Q$$

Hence $f'(x)$ has the factor $(x-r)^{m-1}$. If it had the factor $(x-r)^m$, then $Q(x)$ would have the factor $x-r$, contrary to $Q(r)\neq 0$. We may state our conclusion as follows.

THEOREM 6. *Any multiple root of* $f(x)=0$ *of multiplicity* $m>1$ *is a root of* $f'(x)=0$ *of multiplicity* $m-1$. *A simple root of* $f(x)=0$ *is not a root of* $f'(x)=0$.

THEOREM 7. *If* $f(x)=0$ *and* $f'(x)=0$ *have a common root* r, *which is a root of* $f'(x)=0$ *of multiplicity* $m-1$, *then* r *is a root of* $f(x)=0$ *of multiplicity* m.

Proof. Let k be the multiplicity of the root r of $f(x)=0$. Theorem 6 shows that $k>1$ and that r is a root of $f'(x)=0$ of multiplicity $k-1$. Thus

Let $r_1, \cdots, r_s$ be all the multiple roots of $f(x)=0$, while $r_{s+1}, \cdots, r_t$ are all its simple roots. Let $m_1, \cdots, m_s$ be the multiplicities ($\geqq 2$) of $r_1, \cdots, r_s$. Then $m_1-1, \cdots, m_s-1$ are their multiplicities for $f'(x)=0$. Then

$$(15) \qquad G(x)=(x-r_1)^{m_1-1}\cdots(x-r_s)^{m_s-1}$$

is an exact divisor of both $f(x)$ and $f'(x)$. But if $i>s$, $x-r_i$ is not a divisor of both. The product derived from (15) by increasing any exponent is not

a divisor of $f'(x)$. Hence $G(x)$ is a greatest common divisor (g.c.d.) of $f(x)$ and $f'(x)$. Of course the product of G by any constant $\neq 0$ is another g.c.d. This discussion leads to the following results.

THEOREM 8. *If* $f(x)=0$ *and* $f'(x)=0$ *have at least one common root, then* $f(x)$ *and* $f'(x)$ *have a greatest common divisor* $G(x)$, *which actually involves* x. *A root of* $G(x)=0$ *of multiplicity* $m-1$ *is a multiple root of* $f(x)$ *of multiplicity* m. *Conversely, any multiple root of* $f(x)=0$ *of multiplicity* m $(m \geqq 2)$ *is a root of* $G(x)=0$ *of multiplicity* $m-1$. *If* $q(x)$ *denotes the quotient of* $f(x)$ *by* $G(x)$, *the roots of* $q(x)=0$ *coincide with the distinct roots of* $f(x)=0$.

In practice we do not first find the roots $r_1, \cdots, r_s$ and then compute G by (15), but we proceed in the reverse order, as explained in the following examples (cf. § 57).

EXAMPLE 1. Given $f(x)=16x^4-24x^2+16x-3$, find $G(x)$.

Solution. We have $f'=16(4x^3-3x+1)$. Using "long division" (§ 9), we divide * $4f$ by f' and obtain the quotient x and remainder $-12h$, where $h=4x^2-4x+1$. Next, we divide f' by h and obtain the quotient $16(x+1)$ and remainder zero. Hence we have

Thus h is a g.c.d. of f and f'. Since $h=(2x-1)^2$, $\frac{1}{2}$ is a double root of $G=h=0$ and hence is a triple root of $f=0$. If r denotes the missing root, $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+r=0$ (§ 16), and $r=-\frac{3}{2}$ is a simple root of $f=0$.

EXAMPLE 2. Test $f(x)=x^3-2x^2-4x+8=0$ for multiple roots.

Solution. Here $f'(x)=3x^2-4x-4$. By division we get

$$)-32(x-2).$$

But $x-2$ is a factor of $f'(x)$. Hence $x-2$ is a g.c.d. of $f(x)$ and $f'(x)$. Thus 2 is a double root of $f(x)=0$. Since the sum of its three roots is 2, the remaining root is -2 and it is a simple root.

PROBLEMS

Find the double roots of

1. $x^3-4x^2-35x+150=0$.
2. $x^3-7x^2+15x-9=0$, *Ans.* 3.
3. $x^4-2x^3-39x^2+40x+400=0$.
4. $x^4-8x^2+16=0$, *Ans.* ± 2.
5. $x^3-4x^2-16x+64=0$.
6. $x^5+10x^4+25x^3-2x^2-20x-50=0$.

* The division of f itself by f' introduces fractions.

Find the triple roots of

7. $x^4+10x^3+24x^2-32x-128=0$.
8. $x^4-6x^2-8x-3=0$. *Ans.* -1.
9. $x^5-12x^4+46x^3-40x^2-96x+128=0$.
10. $(x^2-4)^3=0$.

Test for multiple roots

11. $x^3-4x^2-3x+18=0$.
12. $x^4-8x^3+22x^2-24x+9=0$, *Ans.* 1, 1, 3, 3
13. $x^4+5x^3+6x^2-4x-8=0$.
14. $x^3-6x^2+11x-6=0$, *Ans.* None.
15. $x^4-24x^2-64x-48=0$.
16. $x^4-9x^3+9x^2+81x-162=0$, *Ans.* D.R.3.
17. $x^4-4x^3+2x^2+4x+1=0$.
18. $x^4-8x^3+10x^2+24x+9=0$.
19. $x^4+x^3-9x^2+11x-4=0$.
20. $x^4-4x^2+4x-1=0$.
21. $8x^4-20x^3+18x^2-7x+1=0$.
22. $4x^5+8x^4-23x^3-19x^2+55x-25=0$.

50. Horizontal Tangents. If (x, y) is a bend point of the graph of $y=f(x)$, the slope of the tangent at (x, y) is zero by the definition of a bend point. We saw at the end of § 45 that this slope is $f'(x)$. Hence the abscissa x of a bend point is a root of $f'(x)=0$.

In Problems 4–11 and the example in § 45, it was true, conversely, that any real root of $f'(x)=0$ is the abscissa of a bend point. However, this is not always the case. We shall consider in detail an example illustrating this fact. The example is the one merely mentioned in § 44 to indicate the need of the second requirement made in our definition of a bend point.

The graph (Fig. 13) of $y=x^3$ has no bend point since x^3 increases when x increases. Nevertheless, the derivative $3x^2$ of x^3 is zero for the real value $x=0$. The tangent to the curve at $(0, 0)$ is the horizontal line $y=0$. It may be thought of as the limiting position of a secant through O which meets the curve in two further points, seen to be equidistant from O. When one, and hence also the other, of the latter points approaches O, the secant approaches the position of tangency. In this sense the tangent at O is said to meet the curve in three coincident points, their abscissas being the three coinciding roots of $x^3=0$. It is the oddness of the multiplicity of the root $x=0$ which accounts for the fact that $(0, 0)$ is not a bend point. This statement will become clear after we have developed the general theory which follows. This example was given in advance to indicate the main purpose of that theory.

51. Ordinary and Inflexion Tangents. The tangent to the graph of $y=f(x)$ at the point (a, b) on it has the slope $f'(a)$, so that the equation of the tangent is

$$y-b=f'(a)(x-a), \tag{16}$$

In Taylor's formula (7) replace x by a, and h by $x-a$. We get

(17) $$f(x)=f(a)+f'(a)(x-$$

$$(a)\frac{(x-\iota,}{m!}+$$

From the latter and (16) we conclude that the abscissas x of the points of intersection of the graph of $y=f(x)$ with its tangent satisfy the equation

(18) $$f''(a)\frac{(x-a)^2}{1\cdot 2}+\cdots+f^{(m)}(a)\frac{(x-a)^m}{m!}+\cdots=0.$$

The point (a, b) is counted as m coincident points of intersection of the graph and its tangent (just as in the case of $y=x^3$ and its tangent $y=0$ in § 50), if a is a root of multiplicity m of equation (18), and hence if its left member is divisible by $(x-a)^m$, but not by $(x-a)^{m+1}$. This will be true evidently if and only if

(19) $$f''(a)=0,\ \cdots,\quad f^{(m-1)}(a)=0,\quad f^{(m)}(a)\neq 0,$$

in which $m\geqq 2$. When $m=2$, it is to be understood that (19) reduces to the single relation $f''(a)\neq 0$, since this is then the only condition that (18) be divisible by $(x-a)^2$, but not by $(x-a)^3$.

Given $f(x)$ and a, we can readily find the value of m for which relations (19) hold.

For example, if $f(x)=x^4$ and $a=0$, then $f''(0)=f'''(0)=0$, $f^{(4)}(0)=24\neq 0$, so that $m=4$. The graph of $y=x^4$ is a U-shaped curve, whose intersection with the tangent (the x-axis) at (0, 0) is counted as four coincident points of intersection.

Theorem 9. *Determine* m *so that relations* (19) *hold. If* m *is even, the points of the graph of* y=f(x) *in the vicinity of the point of tangency* (a, b) *are all on the same side of the tangent, which is then called an* ***ordinary tangent.*** *But if* m *is odd, the graph crosses the tangent at the point of tangency* (a, b), *and this point is called an* ***inflexion point,*** *while the tangent is called an* ***inflexion tangent.***

For example, in Fig. 13, OX is an inflexion tangent, while the tangent at any point except O is an ordinary tangent. In each of the later Figs. 18, 19, 20, the tangent at

the point whose abscissa is zero is an inflection tangent and all other tangents are ordinary tangents.

Proof. The ordinate of the point of the graph $y=f(x)$ having the abscissa x will be denoted by Y to distinguish it from the ordinate y of the corresponding point of the tangent. Thus Y has the value in (17). From (16) and (17) we see by subtraction that $Y-y$ has the value in (18). Omitting terms which are zero by (19), we get

$$(20)\qquad Y-y=c(x-a)^m+d(x-a)^{m+1}+\cdots,\quad c=\frac{f^{(m)}(a)}{m!},\quad d=\frac{f^{(m+1)}(a)}{(m+1)!},$$

while $c\neq 0$. When $x-a$ is sufficiently small numerically, Theorem 2 of § 46, with $h=x-a$, shows that the sum of the terms after $c(x-a)^m$ is numerically less than $p(x-a)^m$, whatever positive value independent of x we assign to p. We take p less than the numerical value of c. Then (20) shows that $Y-y$ has the same sign as $c(x-a)^m$ for all values of x sufficiently close to a, whether $x>a$ or $x<a$. Hence if m is even, all points on the graph in the vicinity of the point of tangency (a, b) are on the same side of the tangent. But if m is odd, all points on the graph for which $x-a$ is positive and small lie on one side of the tangent, and those for which $x-a$ is negative and numerically small lie on the opposite side. This proves Theorem 9.

52. Criterion for Bend Points. By Theorem 9, a is the abscissa of an inflexion point of the graph of $y=f(x)$ if and only if conditions (19) hold with m odd ($m\geqq 3$). In the theory of equations we are primarily interested in the abscissas a of only those points of inflexion whose inflexion tangents are horizontal, and are interested in them because we must exclude such roots a of $f'(x)=0$ when seeking the abscissas of bend points, which are the important points for our purposes. A point on the graph at which the tangent is both horizontal and an ordinary tangent is a bend point by the definition in § 44. Hence if we apply Theorem 9 to the special case $f'(a)=0$, we obtain the following criterion.

THEOREM 10. *Any root* a *of* $f'(x)=0$ *is the abscissa of a bend point of the graph of* $y=f(x)$ *or of a point with a horizontal inflexion tangent according as the value of* m *for which relations* (19) *hold is even or odd.*

For example, if $f(x)=x^4$, then $a=0$ and $m=4$, so that (0, 0) is a bend point of the U-shaped graph of $y=x^4$. If $f(x)=x^3$, then $a=0$ and $m=3$, so that (0, 0) is a point with a horizontal inflection tangent (OX in Fig. 13) of the graph of $y=x^3$.

PROBLEMS

1. If $f(x)=3x^5+5x^3+4$, the only real root of $f'(x)=0$ is $x=0$. Show that (0, 4) is an inflexion point, and thus that there is no bend point and hence that $f(x)=0$ has a single real root.

2. Prove that $y=x^3-3x^2+3x+c$ has a horizontal inflexion tangent, but no bend point.

3. Show that $y=x^5-10x^3-20x^2-15x+c$ has two bend points and no horizontal inflexion tangents. Use Problem 8 of the preceding set.

4. Prove that $y=3x^5-40x^3+240x+c$ has no bend point, but has two horizontal inflection tangents. Use Problem 4 of the preceding set.

5. Prove that $y=4x^5+25x^4+40x^3-40x^2-160x+c$ has just two bend points $(1, c-131)$ and $(-2, c+112)$, but no horizontal inflexion tangent. There are exactly three real roots if c lies between -112 and 131; otherwise exactly one real root.

6. Show that $y=3x^4-8x^3-24x^2+96x+c$ has the single bend point $(-2, c-176)$, and a single horizontal inflexion tangent at $(2, c+80)$. There are exactly two real roots if $c<176$; otherwise none.

Discuss similarly

7. $y=x^5-40x^3-160x^2-240x+c$. **8.** $y=3x^4-4x^3$

9. $y=3x^4-28x^3+90x^2-108x+c$. **10.** y

11. Prove that any function $x^3-3\alpha x^2+\cdots$ of the third degree can be written in the form $f(x)=(x-\alpha)^3+ax+b$. The straight line having the equation $y=ax+b$ meets the graph of $y=f(x)$ in three coincident points with the abscissa α and hence is an inflexion tangent. If we take new axes of coordinates parallel to the old and intersecting at the new origin $(\alpha, 0)$, i.e., if we make the transformation $x=X+\alpha$, $y=Y$ of coordinates, we see that the equation $f(x)=0$ becomes a reduced cubic equation $^3+pX+q=0$ (§ 33)

12. Find the inflexion tangent to $y=x^3+6x^2-3x+1$ and transform $x^3+6x^2-3x+1=0$ into a reduced cubic equation. *Ans.* $y=-15x-7$, $X^3-15X+23=0$.

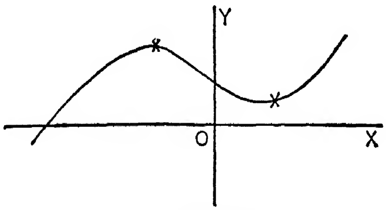
53. Real Roots of a Real Cubic Equation. It suffices to consider

Then $f'=3(x^2-l)$, $f''=6x$. If $l<0$, there is no bend point and the cubic equation $f(x)=0$ has a single real root.

If $l>0$, there are two bend points

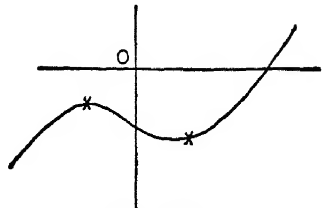
$$(\sqrt{l},\quad q-2l\sqrt{l}),\quad (-\sqrt{l},\quad q+2l\sqrt{l}),$$

which are shown by crosses in Figs. 18–20 for the graph of $y=f(x)$ in the three possible cases specified by the inequalities shown below the figures.



$q \geqq 2l\sqrt{l}$

FIG. 18



$q \leqq -2l\sqrt{l}$

FIG. 19

For a large positive x, the term x^3 in $f(x)$ predominates, so that the graph contains a point high up in the first quadrant, thence extends downward to the right-hand bend point, then ascends to the left-hand bend point, and finally descends. As a check, the graph contains a point far down in the third quadrant, since for x negative, but sufficiently large numerically, the term x^3 predominates and the sign of y is negative.

$2l\sqrt{l} < q < 2l\sqrt{l}$

FIG. 20

If the equality sign holds in Fig. 18 or Fig. 19, a necessary and sufficient condition for which is $q^2=4l^3$, one of the bend points is on the x-axis, and the cubic equation has a double root. The inequalities in Fig. 20 hold if and only if $q^2<4l^3$, which implies that $l>0$.

THEOREM 11. *The equation* $x^3-3lx+q=0$ *has three distinct real roots if and only if* $q^2<4l^3$, *a single real root if and only if* $q^2>4l^3$, *a double root (necessarily real) if and only if* $q^2=4l^3$ *and* $l\neq 0$, *and a triple root if* $q^2=4l^3=0$.

PROBLEMS

Find the bend points, sketch the graph, and find the number of real roots of

1. $x^3+8x+32=0$.
2. $x^3-7x+7=0$, *Ans.* $(\pm\sqrt{\frac{7}{3}},\ 7\mp\frac{14}{3}\sqrt{\frac{7}{3}})$, three.
3. $x^3-6x-6=0$.
4. $x^3-2x-1=0$, *Ans.* $(\pm\sqrt{\frac{2}{3}},\ -1\mp\frac{4}{3}\sqrt{\frac{2}{3}})$, three.
5.

6. $x^3+6x^2-3x+1=0$, *Ans.* $(-2\pm\sqrt{5},\ 23\mp10\sqrt{5})$, one.
7. $x^3-6x^2-4=0$.
8. $x^3-9x-12=0$, *Ans.* $(\pm\sqrt{3},\ \mp6\sqrt{3}-12)$, one.
9. $x^3-3x+1=0$.
10. $3x^4-8x^3+6x^2-24x-12=0$, *Ans.* $(2,\ -52)$, two.

Prove that there are only two real roots of

11. $x^4+3x^2-6x-2=0$. **12.** $x^4+2x^2+8x-3=0$.
13. $x^4+3x^2-10x-6=0$. **14.** $x^4+4x^2-8x-4=0$.

15. Prove that the inflexion point of $y=x^3-3lx+q$ is $(0,\ q)$.

16. Show that Theorem 11 is equivalent to that in § 36.

17. Prove that, if m and n are positive odd integers and $m>n$, $x^m+px^n+q=0$ has no bend point and hence has a single real root if $p>0$; but, if $p<0$, it has just two bend points which are on the same side or opposite sides of the x-axis according as

$$\left.\frac{np}{m}\right)^{m}+\left(\frac{nq}{m-n}\right.$$

is positive or negative, so that the number of real roots is 1 or 3 in the respective cases.

18. Prove that, if p and q are positive, $x^{2m}-px^{2n}+q=0$ has four distinct real roots, two pairs of equal roots, or no real root, according as

$$\left.\frac{nq}{m-n}\right)^{m-n}>0, \quad =0, \quad \text{or} \quad <0.$$

19. Prove that no straight line crosses the graph of $y=f(x)$ in more than n points if the degree n of the real polynomial $f(x)$ exceeds unity. [Apply Theorems 2 and 6 of Chapter II.] This fact serves as a check on the accuracy of a graph.

CHAPTER VII

Number of Real Roots; Isolation of a Root

54. Rolle's Theorem. *Between two consecutive real roots* a *and* b *of* f(x) = 0, *there is an odd number of real roots of* f′(x) = 0, *a root of multiplicity* m *being counted* m *times.*

For example, in Fig. 20 the abscissas of the (bend) points marked with crosses are the two roots of $f'(x)=0$. The right-hand one lies between the two positive roots of $f(x)=0$. The left-hand one lies between the negative root and the smaller positive root.

Proof. Let

where $Q(x)$ is a polynomial divisible by neither $x-a$ nor $x-b$. Then by the rule for the derivative of a product (IV of § 45), we see that

$$\frac{(x-a)(x-b)f'(x)}{} = r(x \qquad a)(x-b)\,Q(x)$$

The second member has the value $r(a-b)<0$ for $x=a$ and the value $s(b-a)>0$ for $x=b$, and hence vanishes an odd number of times between a and b (§ 47). But, in the left member, each of $(x-a)(x-b)$ and $f(x)$ remains of constant sign between a and b, since $f(x)=0$ has no root between a and b. Hence $f'(x)$ vanishes an odd number of times between a and b.

Corollary. *Between two consecutive real roots α and β of* f′(x) = 0 *there occurs at most one real root of* f(x) = 0.

Proof. If there were two such real roots a and b of $f(x)=0$, the theorem shows that $f'(x)=0$ would have a real root between a and b and hence between α and β, contrary to hypothesis.

Applying also § 47 we obtain the

Criterion. *If α and β are consecutive real roots of* f′(x) = 0, *then* f(x) = 0 *has a single real root between α and β if* f(α) *and* f(β) *have opposite signs,*

but no root if they have like signs. At most one real root of f(x) = 0 *is greater than the greatest real root of* f′(x) = 0, *and at most one real root of* f(x) = 0 *is less than the least real root of* f′(x) = 0.

If $f(\alpha)=0$ for our root α of $f'(x)=0$, α is a multiple root of $f(x)=0$ and it would be removed before the criterion is applied.

EXAMPLE. For $f(x)=3x^5-25x^3+60x-20$,

Hence the roots of $f'(x)=0$ are ± 1, ± 2. Now

$$f(-\infty)=-\infty,\quad f(-2)=-36,\quad f(-1)=-58,\quad f(1)=18,\quad f(2)=-4,\quad f(\infty)=\infty.$$

Hence there is a single (positive) real root in each of the intervals

$$(-1,\ 1),\qquad (1,\ 2),\qquad (2,+\infty),$$

and no further real roots. Let k, l, m denote the real roots. Let $q(x)$ denote the quotient of $f(x)$ by $(x-k)(x-l)(x-m)$. The roots of $q(x)=0$ are roots of $f(x)=0$ and are distinct from k, l, m. Hence the former roots are imaginary.

PROBLEMS

1. Prove that $x^5-5x+2=0$ has 1 negative, 2 positive, and 2 imaginary roots.
2. Prove that $x^6+x-1=0$ has 1 negative, 1 positive, and 4 imaginary roots.
3. Show that $x^5-3x^3+2x^2-5=0$ has two imaginary roots, and a real root in each of the intervals $(-2,\ -1.5)$, $(-1.5,\ -1)$, $(1,\ 2)$.
4. Prove that $4x^5-3x^4-2x^2+4x-10=0$ has a single real root.

Find intervals in which the real roots lie for

5. $3x^4-8x^3-24x^2+96x+1=0$. 6. $3x^4-4x^3-$
7. $x^5-10x^3-20x^2-15x+1=0$. 8.

9. Show that, if $f^{(k)}(x)=0$ has imaginary roots, $f(x)=0$ has imaginary roots.
10. Derive Rolle's theorem from the fact that there is an odd number of bend points between a and b, the abscissa of each being a root of $f'(x)=0$ of odd multiplicity, while the abscissa of an inflexion point with a horizontal tangent is a root of $f'(x)=0$ of even multiplicity.

55. Descartes' Rule of Signs. Consider a real polynomial or equation from which we have suppressed all terms having zeros as coefficients. Then two consecutive terms are said to present a *variation of sign* if their coefficients have unlike signs. For example, the first two terms of $x^5-2x^3-4x^2+3$ present a variation of sign, and likewise the last two terms.

Descartes' Rule. *The number of positive real roots of a real equation either is equal to the number* v *of its variations of sign or is less than* v *by a positive even integer. A root of multiplicity* m *is here counted* m *times.*

For example, $x^6-3x^2+x+1=0$ has either two positive roots or none (since $v=2$), the exact number not being found. The two positive roots may coincide and give a double root; if they are distinct, neither is a multiple root. But $3x^3-x-1=0$ has exactly one positive root, which is not a multiple root.

If the rule is true for an equation $f(x)=0$, it is evidently true for $x^t f=0$ and also for $-f=0$ (since the variation of sign for 3, -2 implies one for -3, $+2$). Hence it remains only to prove the rule for

$$f(x) = a_{(} \qquad\qquad a_0>0,$$

in which some of $\cdot$, a_{n-1} may be zero.

Lemma 1. *If* p *is a positive real number and if* (x−p)f(x) *is equal to the polynomial* F(x), *the number of variations of sign of* F(x) *is equal to that of* f(x) *increased by a positive odd integer.*

For example, let the coefficients of $f(x)$ be all different from zero and have the signs in the first line of the following scheme:

$$\begin{array}{rl} xf: & +++----+++-- \\ -pf: & \quad ---++++---++ \\ xf-pf: & \qquad\qquad\qquad\quad -\pm+ \end{array}$$

The first four signs in the third line present a single variation of sign except in the case $+-+-$, when there are 3 variations of sign. In general, any succession of signs, the last of which is opposite to the first, presents an odd number of variations of sign. The further such successions in the third line of the scheme are $-\pm\pm\pm+$, $+\pm\pm-$, and $-\pm+$. Hence the number of variations of sign in the third line is the sum of four positive odd numbers. This sum is $4+e$, where e is an even integer $\geqq 0$. The number of variations of sign of $f(x)$ is 3. That for $F(x)$ is $4+e=3+1+e$, and $1+e$ is a positive odd integer.

To give a general proof of the lemma, let a_{k_1} be the first negative coefficient of $f(x)$, let a_{k_2} be the first positive coefficient following a_{k_1}, let a_{k_3} be the first negative coefficient following a_{k_2}, etc. Finally, for $j=k_v$, $a_j \neq 0$, while each of $a_j, a_{j+1}, \cdots, a_n$ is either zero or is of the same sign as a_n, and the sign of a_j is opposite to that of the coefficient having the subscript k_{v-1}. In the preceding example, $k_1=3$, $k_2=7$, $k_3=10$, $v=3$.

Clearly variations of sign in $a_0, \cdots, a_n$ arise only for two consecutive terms the second of which is one of $a_{k_1}, a_{k_2}, \cdots, a_{k_v}$, whence v is the number of variations of sign of $f(x)$.

Let p be a positive real number. By actual multiplication,

(1) $$F(x)=(x-p)f(x)=P_0x^{n+1}+P_1x^n+\cdots+P_nx+P_{n+1},$$

where

(2) $$P_0=a_0,\ P_1=a_1-pa_0,\ P_2=a_2-pa_1,\ \cdots,\ P_n=a_n-pa_{n-1},$$
$$P_{n+1}=-pa_n.$$

We shall prove that the numbers

(3) $$P_0,\ P_{k_1},\ P_{k_2},\ \cdots,\ P_{k_v},\ P_{n+1}$$

are all different from zero and have the same signs as

(4) $$a_0,\ a_{k_1},\ \cdots,\ a_{k_v},\ -a_n$$

respectively. This is obviously true for P_0 and P_{n+1} by the first and last equations (2). Next, P_{k_i} is the sum of the non-vanishing number a_{k_i} and the number $-pa_{k_i-1}$; the latter is either zero or else is of the same sign as a_{k_i}, since a_{k_i-1} is either zero or of opposite sign to a_{k_i} by our definitions.

By their definition, the successive numbers (4) alternate in sign. Hence the same is true of the numbers (3). In other words, these numbers (3) present $v+1$ variations of sign.

By interpolating further P's in (3), we may enlarge (3) to the set $P_0, P_1, P_2, \cdots, P_{n+1}$ of all coefficients of (1). We saw in the example that any succession of signs, the last of which is opposite to the first, presents an odd number of variations of sign. Hence each of the $v+1$ sub-sets

$$P_0,\ P_1,\ \cdots,\ P_{k_1};\quad P_{k_1},\ P_{k_1+1},\ \cdots,\ P_{k_2};\quad \cdots;$$
$$P_{k_{v-1}},\ \cdots,\ P_{k_v};\quad P_{k_v},\ P_{k_v+1},\ \cdots,\ P_{n+1}$$

presents an odd number of variations of sign. The total number of variations of sign of $P_0, \cdots, P_{n+1}$ is therefore $v+1+2M$, where M is zero or a positive integer. The number of variations of sign of $f(x)$ was seen to be v. Hence the number of variations of sign of $F(x)$ exceeds that of $f(x)$ by the positive odd integer $1+2M$. This proves Lemma 1.

To prove Descartes' rule, consider first the case in which $f(x)=0$ has no

positive real root, that is, no real root between 0 and ∞. Then $f(0)$ and $f(\infty)$ are of the same sign by § 47, and hence a_n and a_0 are of the same sign by § 48. Thus the number v of variations of sign of $f(x)$ is an even integer $\geqq 0$. Since the number of positive roots is $0=v-v$, Descartes' rule is proved for this case.

Next, let $f(x)=0$ have the positive roots $p_1, \cdots, p_l$ and no other positive root. Since a root of multiplicity m is here counted m times, the p's need not be distinct. Then

$$f(x) \equiv (x-p_1)(x-p_2)\cdots(x-p_l)q(x), \tag{5}$$

where $q(x)$ is a real polynomial such that $q(x)=0$ has no positive root. In the preceding paragraph we saw that the number of variations of sign of $q(x)$ is an even integer $\geqq 0$. By Lemma 1 the number of variations of sign of $(x-p_l)q(x)$ is equal to that of $q(x)$ increased by a positive odd integer. Similarly, when we introduce each new factor $x-p_i$. Hence the number of variations of sign of the final product (5) is equal to that of $q(x)$ increased by the sum of l positive odd integers (each of the form $1+2M$). The latter sum is l plus an even integer $\geqq 0$. We saw that the number of variations of sign of $q(x)$ is an even integer $\geqq 0$. Hence, finally, that of $f(x)$ is l plus an even integer $\geqq 0$. This completes the proof of Descartes' rule.

If $-p$ is a negative root of $f(x)=0$, then p is a positive root of $f(-x)=0$. Hence we obtain the

COROLLARY. *The number of negative roots of* f(x)=0 *either is equal to the number of variations of sign of* f(−x) *or is less than that number by a positive even integer.*

EXAMPLE 1. $f(x)=x^4+3x^3+x-1=0$ has one positive root, one negative root, and two imaginary roots.

Solution. Since $f(x)$ presents just one variation of sign, there is a single positive root p which is not a multiple root. Since $f(-x)=x^4-3x^3-x-1$ presents just one variation of sign, $f(x)=0$ has a single negative root n, not a multiple root. Removing the factors $x-p$ and $x-n$, we obtain a depressed equation (§ 11) of degree 2, whose roots are roots of $f(x)=0$ and hence must be imaginary.

EXAMPLE 2. $f(x)=x^3+x^2+8x+6=0$ has imaginary roots.

Solution. Since $f(-x)$ presents three variations of sign, the corollary does not decide whether $f(x)=0$ has one or three negative roots. To remove this doubt, trans-

pose the terms of odd degree and square both members of $-x^3-8x=x^2+6$. We get $x^6+15x^4+52x^2-36=0$. Replace x^2 by y. We get

which has a single positive root p. A negative or imaginary root y leads to imaginary values of $x=\pm\sqrt{y}$. Hence the only possible real roots of $f(x)=0$ are $-\sqrt{p}$ and $\sqrt{p}$, and the latter positive number is evidently not a root.

PROBLEMS

1. Discuss the real roots of $x^3+9x-6=0$.

2. $x^4+12x^2+5x-9=0$ has one positive root, one negative root, and two imaginary roots.

3. $x^4+a^2x^2+b^2x-c^2=0$ $(c\neq 0)$ has just two imaginary roots.

4. For n even, $x^n-1=0$ has only two real roots.

5. For n odd, $x^n-1=0$ has only one real root.

6. For n odd, $x^n+1=0$ has only one real root.

7. $x^3-2x^2+9x-2=0$ has imaginary roots.

8. $x^5+x^3-x^2+2x-3=0$ has four imaginary roots.

9. $x^3+a^2x^2+b^2=0$ $(b\neq 0)$ has two imaginary roots.

Test for real roots the following equations:

10. $x^4-6x^3+7x^2+6x-2=0$. **11.** $x^4-13x^2+4x+2=0$.

12. $x^4-2x^2+12x-8=0$. **13.** $x^4-x^2+10x-4=0$.

14. In the astronomical problem of three bodies occurs the equation

$$r^5+(3-k)r^4+(3-2k)r^3-kr^2-2kr-k=0,$$

where $0<k<1$. Why is there a single positive real root?

15. If a real equation $f(x)=0$ of degree n has n real roots, the number of positive roots is exactly equal to the number V of variations of sign. Hint: Consider also $f(-x)$.

16. Show that $x^3-x^2+2x+1=0$ has no positive root. Hint: Multiply by $x+1$.

17. Prove that we obtain an upper limit to the number of real roots of $f(x)=0$ between a and b, if we set

$$x=\frac{a+by}{1+y} \qquad \therefore\ y=\frac{x-a}{b-x},$$

multiply by $(1+y)^n$, and apply Descartes' rule to the resulting equation in y. The latter is best found in three steps: $x=b+z$, $z=(a-b)/w$, $w=1+y$, the first and third steps being done by synthetic division (§ 61).

18. Show by the method of Problem 17 that there is a single root between 2 and 4 of $x^3+x^2-17x+15=0$. Here we have $27y^3+3y^2-23y-7=0$.

19. $x^3-2x-5=0$ has a single root between 2 and 3.

Further problems with answers may be chosen from the list of 100 quartic equations in § 39.

56. Isolation of the Real Roots. In the next chapter we shall explain Horner's and Newton's methods of computing the real roots of a given real equation to any assigned number of decimal places. Each such method requires some preliminary information concerning the root to be computed. For example, it would be sufficient to know that the root is between 4 and 5, provided there be no other root between the same limits. But in the contrary case, narrower limits are necessary, such as 4 and 4.3, with the further fact that only one root is between these new limits. Then that root is said to be *isolated.*

We may isolate the real roots of $f(x)=0$ by means of the graph of $y=f(x)$. But to obtain a reliable graph, we saw in Chapter VI that we must employ the bend points, whose abscissas occur among the roots $f'(x)=0$. Since the latter equation is of degree $n-1$ when $f(x)=0$ is of degree n, this method is usually impracticable when n exceeds 3. The method based on Rolle's theorem (§ 54) is open to the same objection.

While Descartes' rule is very easy to apply, it usually fails to give the exact number of all the real roots. When it is used as in Problems 17–19, it gives some (but not complete) information as to the number of roots between a and b.

The most effective method for all such questions is that due to Sturm, which we shall treat next.

57. Sturm's Division Process. Let $f(x)$ be a polynomial with real coefficients. In Examples 1 and 2 of § 49 we explained the usual process for finding a greatest common divisor (g.c.d.) of $f(x)$ and its derivative $f'(x)$. We also explained the use of multipliers (positive constants $c_0, c_1, \cdots$) to avoid the introduction of complicated fractions.

We shall now express this process in general terms. The first step consists in dividing c_0f by f' until we obtain a remainder $r(x)$, whose degree is less than that of f'. If q_1 is the quotient, we have $c_0f\equiv q_1f'+r$. The second step consists in dividing c_1f' by r to obtain a remainder $R(x)$, whose degree is less than that of r. If the new quotient is Q, we have $c_1f'\equiv Qr+R$. The third step consists in dividing r by R, etc.

Sturm (who took $c_0=1$, $c_1=1, \cdots$) modified this process as follows. Employ $f_2=-r$, $f_3=-R, \cdots$ Our second identity becomes $c_1f'\equiv q_2f_2-f_3$, where $q_2=-Q$.

Hence Sturm's process is the following. Let the division of c_0f by f' yield the quotient q_1 and a remainder which becomes f_2 when changed in

sign (the degree of f_2 being less than that of f'). Let the division of c_1f' by f_2 yield the quotient q_2 and a remainder which becomes f_3 when changed in sign. Next divide f_2 by f_3, etc. Thus

$$(6)\qquad c_0f\equiv q_1f'-f_2,\ c_1f'\equiv q_2f_2-f_3,\ c_2f_2\equiv q_3f_3-f_4,\ \cdots,$$

$$c_{k-2}f_{k-2}\equiv q_{k-1}f_{k-1}-f_k.$$

EXAMPLE 1. Apply the process to $f=16x^4-24x^2+16x-3$.

Solution. Since this is Ex. 1 of § 49, we have

$$f'=16(4x^3-3x+1),\quad c_0=4,\quad c_1=3,$$

$$4f\equiv xf'-f_2,\quad f_2=12(4x^2-4x+1),\quad 3f'\equiv 4(x+1)f_2.$$

Hence a g.c.d. of f and f' is f_2 or, if we prefer, $4x^2-4x+1$. The double root $\frac{1}{2}$ of $f_2=0$ is a triple root of $f=0$.

EXAMPLE 2. Apply Sturm's process to $f=x^3+4x^2-7$.

Solution. Here $f'=3x^2+8x$. Taking $c_0=3^2$ and $c_1=32^2$, we get

$$9f\equiv(3x+4)f'-f_2,\qquad f_2=32x+63,$$

$$32^2f'\equiv(96x+67)f_2-f_3,\quad f_3=4221.$$

If f and f' have a g.c.d. G, which is not a constant, the first identity shows that G would divide f_2 and the second identity shows that G would divide f_3; since f_3 is a constant not zero, we have a contradiction.

We shall now explain in general how to choose c_0, c_1, etc. In the process of dividing a polynomial P of degree r by $D=mx^s+\cdots$ (of degree $s<r$) to get finally a remainder whose degree is less than s, we use $r-s+1$ steps, each a multiplication of D followed by a subtraction to obtain a remainder free of x^r, x^{r-1}, $\cdots$, x^s. Hence no fractions will be introduced during the division if we first multiply P by m^{r-s+1} and then divide the product by D. This was the method of selecting the multipliers in Ex. 2. Smaller multipliers often serve, as in Ex. 1.

As in the preceding Exs. 1 and 2, Sturm's process is just as effective as the usual one for finding a g.c.d. of f and f'. In (6), let $-f_k$ be the first constant remainder.

If $f_k\neq 0$, f and f' have no common divisor involving x, since such a divisor would divide f_2, by the first identity (6), then divide f_3, f_4, $\cdots$, f_k.

See Ex. 2. This case arises if and only if $f(x)=0$ has no multiple root (§ 49).

But if $f_k=0$, then f_{k-1} is a g.c.d. of f and f'. First, we see that f_{k-1} is a common divisor of f and f' by using identities (6) in the reverse order. For example, if $f_4=0$, then f_3 divides f_2 (by the third identity), and f_3 divides f' (by the second identity), and finally f_3 divides f (by the first identity). By the preceding paragraph any common divisor of f and f' divides f_{k-1}. Our two results prove that f_{k-1} is a g.c.d. of f and f'. See Ex. 1. The present case arises if and only if $f(x)=0$ has a multiple root.

58. Sturm's Theorem. *Let* $\mathrm{f(x)}$ *be a polynomial with real coefficients such that* $\mathrm{f(x)}=0$ *has no multiple root. Construct the identities** (6), *in which* $\mathrm{f_k}$ *is now a constant* $\neq 0$. *Let* a *and* b *be real numbers neither of which is a root of* $\mathrm{f(x)}=0$, *while* $\mathrm{a}<\mathrm{b}$. *Then the number of real roots between* a *and* b *of* $\mathrm{f(x)}=0$ *is the excess of the number of variations of sign of*

$$f(x),\ f'(x),\ f_2(x),\ \cdots,\ f_{k-1}(x),\ f_k \tag{7}$$

for $\mathrm{x}=\mathrm{a}$ *over the number of their variations of sign for* $\mathrm{x}=\mathrm{b}$. *Terms which vanish are to be discarded before counting the variations of sign.*

Proof. Let $V(x)$ denote the number of variations of sign of the numbers (7).

First, if x_1 and x_2 are real numbers such that no one of the continuous functions (7) vanishes for a value of x between x_1 and x_2 or for $x=x_1$ or for $x=x_2$, the values of any one of these functions for $x=x_1$ and $x=x_2$ are both positive or both negative (§ 47), and therefore $V(x_1)=V(x_2)$.

To illustrate the further theory, let $f(x)=x^3-9x^2+24x-36$. Then

$$f'=3(x-2)(x-4), \qquad 9f\equiv(3x-9)f'-f_2, \quad f_2=18(x+6),$$

$$6f'\equiv(x-12)f_2-f_3, \qquad f_3=-18\times 80.$$

The roots of $f=0$ are 6 and the imaginary roots of the depressed equation $x^2-3x+6=0$. The critical values are 6, the roots 2 and 4 of $f'=0$, and the root -6 of $f_2=0$. By the interval (4, 6) we mean the set of all real numbers which exceed 4 and are less than 6. By the interval (6, ∞) we mean the set of all real numbers exceeding 6. The following table shows the signs of Sturm's function f, f', f_2, f_3 when $x=-\infty$, 0, 3, 5, ∞.

* Usually k is the degree of f, but may be less. If $f=x^3+3bx^2+3b^2x+d$, then $q_1=\frac{1}{3}(x+b)$ and $f_2=b^3-d$ is free of x.

	$(-\infty, -6)$	$(-6, 2)$	$(2, 4)$	$(4, 6)$	$(6, \infty)$
f					
f'					
f_2					
f_3					
$V(x)$					

The discussion preceding the example shows that $V(x)$ has the same value for all x's within any interval. The table shows that $V(x)$ does not change when we pass from the first interval to the second, or from the second to the third, or from the third to the fourth, that is, when we pass over a root of $f_2=0$ or of $f'=0$. But $V(x)$ is reduced by unity when we pass from the fourth to the last interval, that is, when we pass over the (single) real root 6 of $f=0$. These two facts illustrate the second and third steps in the following theory.

Returning to the general $f(x)$, we shall often write f_1 for f' to make the notations uniform.

Second, let R be a root of $f_i(x)=0$, where $1 \leqq i \leqq k$. Identities (6) include

$$c_{i-1} f_{i-1}(x) \equiv q_i f_i(x) - f_{i+1}(x). \tag{8}$$

This identity and all the identities (6) which follow it show that f_{i-1} and f_i have no common divisor involving x, since such a divisor would divide $f_{i+1}, f_{i+2}, \cdots, f_k$, while f_k is a constant $\neq 0$. By hypothesis $f_i(x)$ has the factor $x-R$, so that $f_{i-1}(x)$ does not have this factor. Taking $x=R$ in (8), we get

$$-f_{i+1}(R) = c_{i-1} f_{i-1}(R) \neq 0.$$

Our functions $f_{i-1}(x)$ and $f_{i+1}(x)$ are polynomials and hence are continuous (§ 46), so that each has the same sign for $x=R$ as for $x=R \pm p$, if p is a sufficiently small positive number. Thus the values of

$$f_{i-1}(x), \qquad f_i(x), \qquad f_{i+1}(x)$$

for $x=R-p$ show just one variation of sign (since the first and third values were seen to be of opposite sign), and likewise for $x=R+p$ they show just one variation of sign. In other words, there is no change in the number of variations of sign for these two values of x.

It follows from the first and second cases that $V(s)=V(t)$ if s and t are numbers for neither of which any of the functions (7) vanishes and such that no root of $f(x)=0$ lies between s and t. The last property must be assumed here since our second case excluded the value $i=0$.

Third, let r be a root of $f(x)=0$. By Taylor's formula (§ 45),

If p is a sufficiently small positive number, each of these polynomials in p has the same sign as its first term. For, after removing the factor p, we obtain a quotient of the form $c+F$, where $F=dp+ep^2+\cdots$ is numerically less than c for all values of p sufficiently small (Theorem 2 of § 46), while $c=\mp f'(r)$. Hence in each of the above two formulas, the second member has the same sign as its first term. Thus if $f'(r)$ is positive, then $f(r-p)$ is negative and $f(r+p)$ is positive, so that the functions $f(x)$ and $f_1(x)=f'(x)$ have the respective signs $-\ +$ for $x=r-p$, but have the signs $+\ +$ for $x=r+p$. If $f'(r)$ is negative, their signs are $+\ -$ and $-\ -$, respectively. In each case, $f(x)$, and $f_1(x)$ show one more variation of sign for $x=r-p$ than for $x=r+p$.

For the same p, or a still smaller p (if necessary), no one of the continuous functions $f_1(x), \cdots, f_{k-1}(x)$ vanishes for either $x=r-p$ or $x=r+p$, while $f_1(x)$ does not vanish for any real value of x between $r-p$ and $r+p$. By the corollary in § 54 with $\alpha<r-p$ and $\beta>r+p$, there is at most one real root of $f(x)=0$ which is $\geqq r-p$ and $\leqq r+p$. Thus the root r is the only such root.

Applying also the first and second cases, we conclude that $f_1, \cdots, f_k$ present the same number of variations of sign for $x=r-p$ as for $x=r+p$. We saw that f and f_1 show one more variation of sign for $x=r-p$ than for $x=r+p$. Hence for the entire series of functions (7), we have

$$V(r-p)-V(r+p)=1. \tag{9}$$

If r and s are any two consecutive real roots of $f(x)=0$, the set of all real numbers between r and s will be called an interval. Hence the real roots between a and b of $f=0$ determine certain such intervals. By the result preceding the third case, $V(x)$ has the same value for all numbers x in the same interval. By this fact and the result (9), the value of $V(x)$ in any interval exceeds the value for the next interval by unity. Hence $V(a)$ exceeds $V(b)$ by the number of real roots of $f(x)=0$ between a and b. This proves the theorem.

Example. Isolate the real roots of $x^3+4x^2-7=0$.

Solution. We employ the material in the earlier Ex. 2. For $x=1$, the signs of f, f', f_2, f_3 are $- + + +$, which present a single variation of sign. For $x=2$, the signs are $+ + + +$, which present no variation of sign. Our theorem states that there is a single real root between 1 and 2. For $x=-2$, the signs are $+ - - +$, with two variations. For $x=-1$, the signs are $- - + +$, with one variation. Hence there is a single real root between -2 and -1. The missing root r can be isolated by using $f(x)$ alone. Since $f(-\infty)$ is negative, while $f(-2)$ was seen to be positive, r lies between $-\infty$ and -2. Since $f(-3)$ is positive and $f(-4)$ is negative, r lies between -4 and -3. The fact that there is one and only one real root between -4 and -3 will be expressed in the answers to problems by the notation $(-4, -3)$.

PROBLEMS

Isolate by Sturm's theorem all the real roots of

1. $x^3-2x^2+9x-2=0$.
2. $x^3+2x+20=0$, *Ans.* $(-3, -2)$.
3.
4. $x^3+x^2-2x-1=0$.
5. $x^3-x-9=$
6. $x^3-3x^2-2x+5=0$, *Ans.* $(1, 2)$, $(3, 4)$, $(-2, -1)$.
7. $x^3-15x-30=0$.
8. $x^3+21x-42=0$.
9. $x^3+12x+12=0$.
10. $x^3-7x+7=0$, *Ans.* $(1, 1\frac{1}{2})$, $(1\frac{1}{2}, 2)$, $(-4, -3)$.
11. $x^4-x^2+10x-4=0$.
12. $3x^4-6x^2+8x-3=0$, *Ans.* $(-2, -1)$, $(0, 1)$.
13. $x^5-5x-2=0$.
14. $x^4-8x^3+25x^2-36x+8=0$, *Ans.* $(0, 1)$, $(3, 4)$.
15. $x^4-3x^2-10x-6=0$.
16. $x^4+12x^2+5x-9=0$, *Ans.* $(0, 1)$, $(-2, -1)$.
17. $x^4-8x^2-16x+12=0$.
18. $x^4+12x-5=0$.
19. $x^4-2x^2-8x-3=0$.
20. $x^4-2x^2+12x-8=0$.
21. $x^4+32x-60=0$.
22. $x^4-4x^2+8x-4=0$.
23. If Δ is the discriminant (§ 35) of $f=x^3+px+q=0$ and if $p\neq 0$, show that $f_2=-2px-3q$, $4p^2f'\equiv(-6px+9q)f_2-\Delta$. Prove by Sturm's theorem that there is a single real root if Δ is negative and three distinct real roots if Δ is positive.

59. Device to Shorten the Work by Sturm's Theorem. *When the Sturm's function of the second degree has a negative discriminant, we may replace that function by its first coefficient and discard all later Sturm's functions.*

The chances are therefore even that an equation to be treated by Sturm's method is such that we can make these simplifications.

This theorem is derived at once from the following two lemmas.

LEMMA 2. *Denote by* g(x) *that one of Sturm's functions which is of the second degree. If* g(x) *has the same sign* ($\pm$) *for all real values of* x, *we may replace* g(x) *by* ± 1 *and discard all later Sturm's functions.*

Proof. Let $h(x)$ be the Sturm's function which follows g. First, let h be a constant (necessarily $\neq 0$). Then g and h evidently present the same number of variations of sign for all real values of x. Hence we may discard h from Sturm's functions and replace g by ± 1.

Second, let $h(x)$ be not a constant. Then $h(x)$ is of the first degree in x and we may write $h(x) = d(x-e)$, where $d \neq 0$. By the remainder theorem, $g(x) \equiv (x-e)L + g(e)$, where L is a linear function of x. Hence $g(x) \equiv d(x-e)(L/d) + g(e)$. Hence Sturm's function following $h(x)$ is $-g(e)$, whose sign is $\mp$. Irrespective of the sign of $h(x)$, the functions $g(x)$, $h(x)$, $-g(e)$ present just one variation of sign for every real value of x (since the outer signs are $\pm$ and $\mp$). Hence we may discard $h(x)$ and $-g(e)$ from Sturm's functions and replace $g(x)$ by ± 1.

LEMMA 3. *The real function* $g(x) = ax^2 + bx + c$ *has always the same sign (in fact, the sign of* a), *if and only if its discriminant* $D = b^2 - 4ac$ *is negative.*

Proof. First, if D is negative, the simple identity

$$4ag \equiv (2ax+b)^2 - D \tag{10}$$

shows that ag is always positive, whence g always has the same sign as a.

Second, let $g(x)$ always have the sign $\pm$. By § 48, the sign of $g(\infty)$ is that of a. Hence a has the sign $\pm$. Thus ag is always positive. In identity (10) assign to x the value for which $2ax + b = 0$. We conclude that $-D$ is positive.

EXAMPLE 1. If $f(x) = x^3 + 6x - 10$, then $f' = 3(x^2+2)$ is always positive. Hence we may replace Sturm's functions by f, 1. For $x = -\infty$, there is just one variation of sign; for $x = +\infty$, no variation. Hence there is a single real root. It is seen to lie between 1 and 2.

EXAMPLE 2. If $f(x) = 2x^4 - 13x^2 - 10x - 19$, we have

The discriminant of f_2 is -1751. Since f_2 is therefore always positive, we may replace Sturm's functions by f, f', 1. For $x = -\infty$, their signs are $+ - +$; for $x = +\infty$, $+ + +$. Hence there are exactly two real roots. For $x = 0$, the signs are $- - +$ Hence one root is positive and the other is negative.

PROBLEMS

1. $x^3+3qx^2+3(p+q^2)x+c=0$ has a single real root if $p>0$.

Show that three Sturm's functions suffice to prove that there are exactly two real roots of the following equations.

2. $x^4+4x^3+3x^2-2x-5=0$.

3.

4. $x^4+4x^3+3x^2-2x-8=0$.

5.

6. $x^4+4x^3+3x^2-6x-9=0$.

7.

8. $x^4+bx^3+6x^2+bx+1=0$, $b>4$.

9. $=0$, $b^2>32$.

10. $x^4+bx^3+30x^2+5bx+25=0$, $b^2>80$.

11.

12.

13.

14.

15.

16.

17.

18.

19. $x^4+bx^3+3x^2-E=0$, $b\geqq 4$, $E\geqq 1$.

20. $-E=0$, $E\geqq 2$.

21. $x^4+bx^3+4x^2-E=0$, $b\geqq 5$, $E\geqq 1$.

22.

23. $x^4+bx^3+5x^2-E=0$, $b\geqq 6$, $E\geqq 1$; $b=5$, $E\geqq 2$.

24. $-12ADx+h$,

^2D, $p>0$, $q>0$,

we may stop with Sturm's function

$$f_2=-360(B-2A^2)(x^2+p)(x^2+q).$$

Then if $2A^2>B$ there are exactly two real roots of $f=0$.

Hence prove that the latter is true for

25. $x^6-2x^4-(p+q)x^2-\frac{2}{3}pq=0$.

26. $x^6-6x^5-30x^2+12x-9=0$.

27. Show that an equivalent condition for lemma 3 is that the roots of $g(x)=0$ be imaginary. Give an immediate proof of the modified lemma. Hint: A real root r is excluded since $q(r)=0$.

28. Hence prove that a quartic function $Q(x)$ always has the same sign if and only if all four roots of $Q=0$ are imaginary.*

60. Further Topics. Suppose $f(x)=0$ has multiple roots. As explained in § 57, equations (6) show that f_k is a greatest common divisor of f and f' and hence is now not a constant. Let Q denote the quotient of f by $f_k(x)$. The roots of $Q=0$ coincide with the distinct roots of $f=0$ (§ 49). We may treat $Q=0$ by Sturm's theorem, since it has no multiple root.

However we may modify (*First Course*, page 82) the proof of Sturm's theorem and show that

* The conditions on the coefficients of Q are given in *First Course*. p. 81.

If each multiple root is counted only once, the number of real roots between a *and* b (a<b) *is* V(a) − V(b), *where* V(a) *is the number of variations of sign of* f, f′, f_2, ⋯, f_k(x) *for* x = a.

In Ex. 1 of § 57, these functions may be taken to be f, f', f_2. By use of $x = -\infty, 0, +\infty$, we see that there is a single positive root t and a single negative root of $f = 0$. As before, $t = \frac{1}{2}$ is a triple root.

PROBLEMS

Solve by the last theorem the equations having multiple roots:

1. $x^4 - 4x^3 - 2x^2 + 12x + 9 = 0$, *Ans.* 3, 3, −1, −1.
2. $x^4 + 5x^3 + 9x^2 + 7x + 2 = 0$.
3. $x^4 - x^2 + 2x + 2 = 0$.
4. $x^4 - 2x^3 - 3x^2 + 4x + 4 = 0$, *Ans.* 2, 2, −1, −1.

For twenty-two further suitable problems, see § 49.

Budan's Theorem. *Let* a *and* b (a<b) *be real numbers neither of which is a root of* f(x) = 0, *an equation of degree* n *with real coefficients. Let* V(a) *denote the number of variations of sign of*

$$f(x),\ f'(x),\ f''(x),\ \cdots,\ f^{(n)}(x) \tag{11}$$

for x = a, *after vanishing terms have been discarded. Then the number of real roots of* f(x) = 0 *between* a *and* b *either is* V(a) − V(b) *or is less than that difference by a positive even integer. A root of multiplicity* m *is here counted as* m *roots.*

This theorem rarely gives the exact number of roots. Since a complete proof is quite long (*First Course*, pages 83–85), we shall prove it only in the important case $a = 0$, $b = +\infty$. Let

$$f(x) = a_0x^n + \cdots + a_{n-1}x + a_n = 0$$

For $x = 0$, the functions (11) have the same signs as

Thus $V(0)$ is equal to the number V of variations of sign of $f(x)$. For $x = +\infty$, the functions (11) all have the same sign, which is that of a_0. Thus $V(0) - V(\infty) = V$. By Descartes' rule, the number of positive roots either is V or is less than V by a positive even integer.

Problems may be selected from the earlier sets, especially the long lists in Chapter V.

CHAPTER VIII

Solution of Numerical Equations

61. Horner's Method. After we have isolated a real root of a real equation by one of the methods in Chapter VII, we can compute the root to any desired number of decimal places either by Horner's method, which is available only for polynomial equations, or by Newton's method, which is applicable also to logarithmic, trigonometric, and other equations.

To find the root between 2 and 3 of

$$x^3-2x-5=0, \tag{1}$$

set $x=2+p$. Direct substitution gives the *transformed equation* for p:

(2)

The method just used is laborious especially for equations of high degree. We next explain a simpler method. Since $p=x-2$,

$$x^3-2x-5\equiv(x-2)^3+6(x-2)^2+10(x-2)-1,$$

identically in x. Hence -1 is the remainder obtained when the given polynomial x^3-2x-5 is divided by $x-2$. By inspection, the quotient Q is equal to

$$(x-2)^2+6(x-2)+10.$$

Hence 10 is the remainder obtained when Q is divided by $x-2$. The new quotient is equal to $(x-2)+6$, and another division gives the remainder 6. Hence to find the coefficients 6, 10, -1 of the terms following p^3 in the transformed equation (2), we have only to divide the given polynomial x^3-2x-5 by $x-2$, then divide the quotient Q by $x-2$, etc., and take the remainders in reverse order. However, when this work is performed by synthetic division (§ 10) as tabulated below, no reversal of order is necessary, since the coefficients then appear on the page in their desired order.

1	0	−2	−5
	2	4	4
1	2	2	**−1**
	2	8	
1	4	**10**	
	2		
1	**6**		

Thus 1, 6, 10, −1 are the coefficients of the desired equation (2).

Since $p=x-2$, the roots of equation (2) are obtained by subtracting 2 from each root of equation (1). Hence we may use synthetic division to diminish the roots by 2.

To obtain an approximation to the decimal p, we ignore for the moment the terms involving p^3 and p^2; then by $10p-1=0$, $p=0.1$. But this value is too large since the terms ignored are all positive. For $p=0.09$, the polynomial in (2) is found to be negative, while for $p=0.1$ it was just seen to be positive. Hence $p=0.09+h$, where h is positive and of the denomination thousandths. The coefficients 1, 6.27, $\cdots$ of the transformed equation for h appear in heavy type just under the first zigzag line in the following scheme:

1	6	10	−1	0.09
	0.09	0.5481	0.949329	
1	6.09	10.5481	**−0.050671**	
	0.09	0.5562		
1	6.18	**11.1043**		$\frac{0.05}{11.1}$
	0.09			
1	**6.27**			$=0.004$
	0.004	0.025096	0.044517584	
1	6.274	11.129396	**−0.006153416**	
	0.004	0.025112		
1	6.278	**11.154508**		
	0.004			
1	**6.282**			

Hence $x=2.094+t$, where t is a root of

$$B=0, \quad A=11.154508, \quad C=0.006153416.$$

Denote the part $t^3+6.282t^2$ by either C or $C(t)$. Since C is relatively small, an approximation to t is obtained from $At-B=0$. Evidently B/A lies between $r=0.0005$ and $s=0.0006$. We get

$$C(r)=0.00000157, \qquad C(s)=0.00000226,$$

correct to eight decimal places. Also, $Ar-B=-0.000576$ and $As-B=0.000539$, to six decimal places. Adding $C(r)$ and $C(s)$ to these, respectively, we see that $f(r)$ is negative, and $f(s)$ is positive. Hence the root t lies between r and s.

We may obtain a closer approximation to t as follows. By the definition of C we have $f(t)=At-D$, where $D=B-C$. But $C=C(t)$ lies between $C(r)$ and $C(s)$, whose values were given earlier. Whichever of these two values is chosen, we see that $D=0.006151$, correct to six decimal places. The value of the root $t=D/A$ to six decimal places is found by abridged division as follows.

$$\begin{array}{lrl}
\text{x xx} & & \\
11.154508 & 0.006151 & 0.0005514=t \\
 & 5577 & \\
 & & \\
 & 574 & \\
 & \underline{558} & \\
 & 16 & \\
 & 11 &
\end{array}$$

Since the quotient is $0.0005+$, only two decimal places of the divisor are used, except to see by inspection how much is to be carried when making the first multiplication. Hence we mark a cross above the figure 5 in the hundredths place of the divisor and use only 11.15. Before making the multiplication by the second significant figure 5 of the quotient t, we mark a cross over the figure 1 in the tenths place of the divisor and hence use only 11.1 Thus $x=2.0945514+$, with doubt only as to whether the last figure should be 4 or 5.

If we require a greater number of decimal places, it is not necessary to go back and construct a new transformed equation from the equation in t. We have only to revise our preceding dividend on the basis of our present better value of t. We now know that t is between 0.000551 and

0.000552. To compute the new value of the correction C, in which we may evidently ignore t^3, we use logarithms.

$$\begin{array}{rlrl} & \log\ 5.51 = .74115 & & \log\ 5.52 = .74194 \\ \therefore & \log\ 5.51^2 = 1.48230 & \therefore & \log\ 5.52^2 = 1.48388 \\ & \log\ 6.282 = .79810 & & \log\ 6.282 = .79810 \\ & \log 190.72 = 2.28040 & & \log 191.42 = 2.28198 \end{array}$$

Hence C is between 0.000001907 and 0.000001915. Whichever of the two limits we use, we obtain the same new dividend below correct to eight decimal places.

$$\begin{array}{r|r|l} \overset{\text{x xxxx}}{11.154508} & 0.00615150 & 0.00055148 \\ & 557725 & \\ & 57425 & \\ & \underline{55773} & \\ & 1652 & \\ & 1115 & \\ & 537 & \\ & \underline{446} & \\ & 91 & \\ & 89 & \end{array}$$

Hence, finally, $x=2.094551482$, with doubt only as to the last figure.

To find a negative root of $f(x)=0$, employ $f(-x)=0$.

PROBLEMS

(The number of transformations made by synthetic division should be about half the number of significant figures desired for a root.)

Compute the single real root of the following equations:*

2.

3. $x^3+18x-30=0$, *Ans.* 1.4848066.

5. $x^3-36x-96=0$, *Ans.* 7.0446667.

* For the twenty-two quartic equations in § 59, with numerical values for b, j, m, E, the roots can be isolated quickly by using the three necessary Sturm's functions. Hence those equations may be assigned for solution here.

6. $x^3-18x-42=0.$
7. $x^3+90x-30=0,$ *Ans.* 0.3329234.
8. $x^3-33x-132=0.$
9. $x^3+27x-72=0,$ *Ans.* 2.2466650.
10. $x^3+48x-96=0.$
11. $x^3-27x-90=0,$ *Ans.* 6.4068324.
12. $x^3-42x-126=0.$
13. $x^3+30x-90=0,$ *Ans.* 2.4871541.

Compute the two real roots of

14. $x^4-11727x+40385=0,$ *Ans.* 3.45592, 21.43067.
15. $x^4-16x^2+48x-36=0.$
16. x^4-
17. $x^4-12x^2-40x-21=0,$ *Ans.* $r=4.6457513,\ 4-r.$
18. $x^4-13x^2+44x-28=0.$
19. $x^4+4x^2-24x-20=0,$ *Ans.* $r=2.7320508,\ 2-r.$
20. $x^4+2x^2+28x-40=0.$
21. $x^4-15x^2-36x-20=0,$ *Ans.* $r=4.8284271,\ 4-r.$

Compute the four real roots of

22. $x^4+4x^3-17.5x^2-18x+58.5=0,$ *Ans.* ± 2.1213203, 2.1231056, -6.1231056.
23. x^4
24. x^4
25. x^4
26. x^4
27. x^4
28. x^4
29. $x^4-26x^2+24x+21=0,$ *Ans.* $r=4.4142136,\ s=-5.4494897,\ 6-r,\ -s-6.$

Find the three decimal places the abscissas of the real points of intersection of

30. Parabola $y=x^2$ and hyperbola $xy+x+3y-6=0,$ *Ans.* 1.095.
31. $x^2+y^2=9,\ y=x^2-x,$ *Ans.* 2.059, -1.228.

32. A sphere 2 feet in diameter is formed of a kind of wood a cubic foot of which weighs two-thirds as much as a cubic foot of water (so that the *specific gravity* of the wood is $\frac{2}{3}$). Find to four significant figures the depth h to which the floating sphere will sink in water. *Ans.* 1.226.

Hints. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Hence our sphere whose radius is 1 foot weighs as much as $\frac{4}{3}\pi\cdot\frac{2}{3}$ cubic feet of water. The volume of the submerged portion of the sphere is $\pi h^2(r-\frac{1}{3}h)$ cubic feet. Since this is also the volume of the displaced water, its value for $r=1$ must equal $\frac{4}{3}\pi\cdot\frac{2}{3}$. Hence $h^3-3h^2+\frac{8}{3}=0.$

33. Solve Problem 32 when the specific gravity is $\frac{3}{4}$.

34. If the specific gravity of cork is $\frac{1}{4}$, find to four significant figures how far a cork sphere 2 feet in diameter will sink in water. *Ans.* 0.6527.

35. Compute $\cos 20°$ to four decimal places by use of

$$\cos 3A=4\cos^3 A-3\cos A,\quad \cos 60°=\tfrac{1}{2}.$$

36. Three intersecting edges of a rectangular parallelopiped are of lengths 6, 8 and 10 feet. If the volume is increased by 300 cubic feet by equal elongations of the edges, find the elongation to four decimal places. *Ans.* 1.3500.

37. Solve Problem 36 if the volume is increased by 500 cubic feet.

38. Given that the volume of a right circular cylinder is $\alpha\pi$ and the total area of its surface is $2\beta\pi$, prove that the radius r of its base is a root of $r^3-\beta r+\alpha=0$. If $\alpha=56$

$\beta=28$, find to four decimal places the two positive roots r. The corresponding altitude is α/r^2. *Ans.* 2.7138, 3.3840.

39. What rate of interest is implied in an offer to sell a house for $2700 cash, or in annual installments each of $1000 payable 1, 2, and 3 years from date? *Ans.* 5.46%.

Hint. The amount of $2700 with interest for 3 years should be equal to the sum of the first payment with interest for 2 years, the amount of the second payment with interest for 1 year, and the third payment. Hence if r is the rate of interest and we write x for $1+r$, we have

$$2700x^3=1000x^2+1000x+1000.$$

40. Find the rate of interest implied in an offer to sell a house for $3500 cash, or in annual installments each of $1000 payable 1, 2, 3, and 4 years from date. *Ans.* 5.57%.

41. Find the rate of interest implied in an offer to sell a house for $3500 cash, or $4000 payable in annual installments each of $1000, the first payable now. *Ans.* 9.70%.

42. In a semicircle of diameter x is inscribed a quadrilateral with sides a, b, c, x; then $x^3-(a^2+b^2+c^2)x-2abc=0$ (I. Newton). Given $a=2$, $b=3$, $c=4$, find x to six decimal places. *Ans.* 6.074674.

43. Find x in Problem 42 when $a=3$, $b=4$, $c=5$.

44. What rate of interest is implied in an offer to sell a house for $9000 cash, or $1000 down and $3000 at the end of each year for 3 years? *Ans.* 6.13%.

Using synthetic division, find an equation

45. Whose roots are those of $x^4+x^3-3x^2-x-4=0$ diminished by 2.

46. Whose roots are those of $x^3+x^2-3x+9=0$ increased by 3.

62. Newton's Method. Prior to 1676, Newton had already found the root between 2 and 3 of equation (1). He replaced x by $2+p$ and obtained (2). Since p is a decimal, he neglected the terms in p^3 and p^2, and hence obtained $p=0.1$, approximately. Replacing p by $0.1+q$ in (2), he obtained

$$q^3+6.3q^2+11.23q+0.061=0. \tag{3}$$

Dividing -0.061 by 11.23, he obtained -0.0054 as the approximate value of q. Neglecting q^3 and replacing q by $-0.0054+r$, he obtained

$$6.3r^2+11.16196r+0.000541708=0. \tag{4}$$

Dropping $6.3r^2$, he found r and hence

$$x=2+0.1-0.0054-0.00004853=2.09455147,$$

of which all figures but the last are correct (§ 61). But the method will not often lead so quickly to so accurate a value of the root.

Newton used the close approximation 0.1 to p, in spite of the fact that

this value exceeds the root p and hence led to a negative correction at the next step. This is in contrast with Horner's method in which each correction is positive, so that each approximation must be chosen less than the root, as 0.09 for p.

The systematic computation of the coefficients of (3) and (4) is as follows.

1	6	10	−1	0.1
	0.1	0.61	1.061	
1	6.1	10.61	**0.061**	
	0.1	0.62		
1	6.2	**11.23**		
	0.1			−0.061
1	**6.3**			11.23
	−0.005	−0.031475	−0.0559926	= −0.005
1	6.295	11.198525	**0.0050074**	
	−0.005	−0.031450		
1	6.290	**11.167075**		
	−0.005			−0.005
1	**6.285**			11.167
	−0.0004	−0.002514	−0.0044658	= −0.0004
1	6.2846	11.164561	**0.0005416**	
	−0.0004	−0.002514		
1	6.2842	**11.162047**		
	−0.0004			
1	**6.2838**			

$$r=\frac{-0.0005416}{11.162047}=-0.0000485$$

Hence the root is $2+0.1-0.005-0.0004-0.0000485=2.0945515$, correct to seven decimal places.

We shall present Newton's idea in a useful algebraic form. Let $f(x)$ be a real polynomial. Given an approximate value g of a real root of $f=0$, we can find another approximation $g+h$ to the root by neglecting the powers h^2, h^3, $\cdots$ of the small number h in Taylor's formula (§ 45)

$$f(g+h)=f(g)+f'(g)h+f''(g)\frac{h^2}{2}+\cdots$$

and hence by taking

$$f'(g)$$

We then repeat the process with $g_1 = g+h$ in place of the former g.

Thus in Newton's example, $f(x) = x^3 - 2x - 5$, we have, for $g = 2$,

$$h = \frac{-f(2)}{f'(2)} = \frac{1}{10}, \qquad g_1 = g+h = 2.1,$$

$$h_1 = \frac{-f(2.1)}{f'(2.1)} = \frac{-0.061}{11.23} = -0.0054\cdots.$$

This formulation of Newton's method is applicable also to equations involving logarithmic, trigonometric, or other simple functions. It must be noted that we have proved Taylor's formula only when $f(x)$ is a polynomial, say of degree n. For other functions the second member of Taylor's formula does not stop with the term involving h^n, but contains also like terms involving h^{n+1}, h^{n+2}, $\cdots$ and so on indefinitely. In other words, it is an infinite series and care must be taken to find how small h must be numerically to insure that the series shall converge (to some definite finite value). But such a discussion is beyond the scope of this book.

In view of this inconvenience and the doubt that $g+h$ may give a better approximation than g, we are justified in presenting the rather long treatment given next.

63. Graphical Discussion of Newton's Method. Using rectangular coordinates, consider the graph of $y = f(x)$ and the point P on it with the

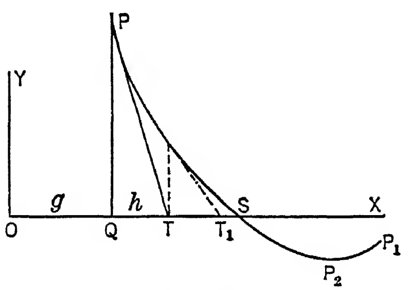


FIG. 21

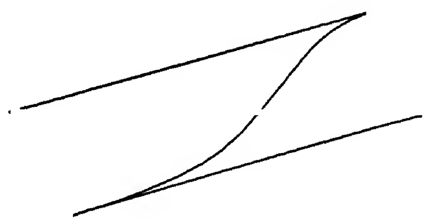


FIG. 22

abscissa $OQ = g$ (Fig. 21). Let the tangent at P meet the x-axis at T and let the graph meet the x-axis at S. Take $h = QT$, the subtangent

Then

$$QP=f(g), \qquad f'(g)=\tan XTP=\frac{-f(g)}{h},$$

$$h=\frac{-f(g)}{f'(g)}.$$

In the graph in Fig. 21, $OT=g+h$ is a better approximation to the root OS than $OQ=g$. The next step (indicated by dotted lines) gives a still better approximation OT_1.

If, however, we had begun with the abscissa g of a point P_2 in Fig. 21 near a bend point, the subtangent would be very large and the method would probably fail to give a better approximation. Failure is certain if we use a point P_1 such that a single bend point lies between it and S.

We are concerned with the approximation to a root previously isolated as the only real root between two given numbers a and b, where $a<b$. These should be chosen so nearly equal that $f'(x)=0$ has no real root between a and b, and hence the graph of $y=f(x)$ has no bend point between a and b. Moreover, if $f''(x)=0$ has a root r between a and b such that $f'''(r)\neq 0$, the graph will have an inflexion point with the abscissa r, and the method will likely fail (Fig. 22). Let, therefore, neither $f'(x)$ nor $f''(x)$ vanish between a and b.

I. Let $f'(x)$ be positive when $a\leqq x\leqq b$. Then the tangent at the point R with the abscissa x makes an acute angle E with the x-axis, since $\tan E=f'(x)>0$. When x increases from a to b, the point R therefore

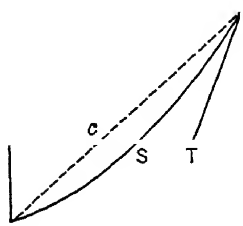


FIG. 23

FIG. 24

moves upwards as it travels along the graph of $y=f(x)$. Thus *if* f′(x) *is positive,* f(x) *increases when* x *increases.* Since the graph ascends, it is like that in Fig. 23 or Fig. 24.

II. Let $f'(x)$ be negative when $a \leqq x \leqq b$. The tangent at the point R with the abscissa x makes an obtuse angle E with the x-axis, since $\tan E = f'(x) < 0$. When x increases from a to b, the point R moves downwards as it travels along the graph of $y = f(x)$. Thus *if* f′(x) *is negative,* f(x) *decreases when* x *increases.* Since the graph descends, it is like that in Fig. 25 or Fig. 26.

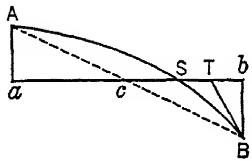


Fig. 25

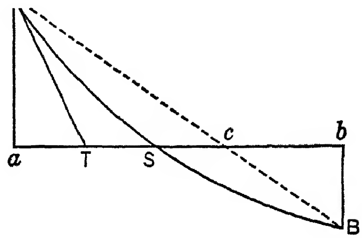


Fig. 26

We shall now subdivide cases I and II as follows.

I_1. Let both $f'(x)$ and $f''(x)$ be positive when $a \leqq x \leqq b$. Apply the result stated in italics in case I with $f(x)$ replaced by the new function $f'(x)$. Since its derivative $f''(x)$ is positive, we conclude that $f'(x)$ increases when x increases. Thus the angle E made by the tangent with the x-axis increases when x increases.* Hence the graph is like that in Fig. 23, and not like that in Fig. 24. We take $g = b$ to be the initial approximation to a root of $f(x) = 0$ by Newton's process. Then the next step in that process yields a better approximation than g to the root. In fact, of the two points T and the point marked b, T is the one which is nearer to S (Fig. 23).

I_2. Let $f'(x)$ be positive and $f''(x)$ be negative when $a \leqq x \leqq b$. Apply the result stated in italics in case II with $f(x)$ replaced by the new function $f'(x)$. Since its derivative $f''(x)$ is negative, we conclude that $f'(x)$ decreases when x increases. The same is therefore true of angle E, so that the graph is like that in Fig. 24. We take $g = a$ in Newton's process, and see that the next step in that process yields a better approximation than g to the root. In fact, of the points T and the point marked a, T is the one which is nearer to S.

II_1. Let $f'(x)$ be negative and $f''(x)$ be positive when $a \leqq x \leqq b$. Since

* If we place a pencil tangent at A to the graph in Fig. 23 and move the pencil so that it remains tangent, we see that the pencil rotates toward a vertical position. But for Fig. 24, the pencil rotates toward a horizontal position.

the derivative of $f'(x)$ is positive, $f'(x)$ increases when x increases. Likewise for angle E. The graph is therefore like that in Fig. 26 (and not like that in Fig. 25). We take $g=a$.

II_2. Let $f'(x)$ and $f''(x)$ both be negative when $a \leqq x \leqq b$. Since the derivative of $f'(x)$ is negative, $f'(x)$ decreases when x increases. Likewise for angle E. The graph is therefore like that in Fig. 25. We take $g=b$.

In both the subcases II_1 and II_2, we see (as in cases I_1 and I_2) that, of the points T and the point on the x-axis having the abscissa g (viz., a or b, respectively), T is the one which is nearer to S. Hence Newton's process again succeeds.

The results in the four subcases may be combined as follows.

THEOREM. *If* f(x) *has a single real root between* a *and* b, *and if neither* f′(x) = 0 *nor* f″(x) = 0 *has a real root between* a *and* b, *and if we designate by* g *that one of the numbers* a *and* b *for which* f(g) *and* f″(g) *have the same sign, then* g − f(g)/f′(g) *is closer to the root than* g.

Let k denote that one of a and b which is not g. We obtain a value c which is closer to the root than k if we take c to be the abscissa of the intersection of the x-axis with the chord AB in Figs. 23, 24, 25, 26. By similar triangles,

$$-f(k) : c-k = f(g) : g-c, \quad \text{or} \quad c= \tag{5}$$

EXAMPLE. $f(x)=x^3-2x^2-2$, $a=2\frac{1}{4}$, $b=2\frac{1}{2}$. Then

$$f(a)=\tfrac{47}{64}, \quad f(b)=\tfrac{9}{8}.$$

Neither of the roots 0 and $\frac{4}{3}$ of $f'(x)=0$ lies between a and b, so that $f(x)=0$ has a single real root between these limits (§ 54). Nor is the root $\frac{2}{3}$ of $f''(x)=0$ within these limits. The conditions of the theorem are therefore satisfied. For $a<x<b$, the graph is of the type in Fig. 23. Hence $g=b$, $k=a$. We find that approximately

$$c=\frac{559}{238}=2.3487, \qquad g_1=g-\frac{f(g)}{f'(g)}=2.3714,$$

$$g_1-\frac{f(g_1)}{f'(g_1)}=2.3597.$$

For $x=2.3594$, $f(x)=0.0007$. For $x=2.3593$, $f(x)=-0.00003$. We therefore have the root to four decimal places. For $m=2.3593$,

$$f'(m)=7.2620, \qquad m-\frac{f(m)}{f'(m)}=2.3593041,$$

which is the value of the root correct to seven decimal places. We at once verify that the result is greater than the root in view of our work and Fig. 23, while if we change the final digit from 1 to 0, $f(x)$ is negative.

PROBLEMS

1. For $f(x) = x^4 + x^3 - 3x^2 - x - 4$, show by Descartes' rule of signs that both $f'(x) = 0$ and $f''(x) = 0$ have a single positive root and that neither has a root between 1 and 2. Which of the values 1 and 2 should be taken as g? *Ans.* 2.

2. When seeking a root between 2 and 3 of $x^3 - x - 9 = 0$, which value should be taken

Find by Newton's method the single real root of

3. Equations in Problems 1, 2.

4. $x^3 - 36x - 84 = 0$, *Ans.* 6.9361683

5. $x^3 = 12$.

6. $x^3 - 60x - 180 = 0$, *Ans.* 8.9504582

7. $x^3 = 26$.

8. $x^3 - 30x - 110 = 0$, *Ans.* 6.7960235

9. $x^3 + 78x - 65 = 0$.

10. $x^3 + 84x - 84 = 0$, *Ans.* 0.9885012

11. $x^3 - 45x - 120 = 0$.

12. $x^3 + 63x - 84 = 0$, *Ans.* 1.2985750

Find the two real roots of

13. $x^4 - 5x^2 + 22x - 30 = 0$.

14. $x^4 + x^2 + 30x - 50 = 0$.

15. $x^4 - 17x^2 + 44x - 30 = 0$.

16. $x^4 - 10x^2 + 40x - 16 = 0$.

17. $x^4 - 11x^2 - 44x - 24 = 0$, *Ans.* $r = 4.6457513$, $4 - r$.

18. $x^4 - 14x^2 + 56x - 48 = 0$.

19. $x^4 -$

64. Newton's Method for Trigonometric and Logarithmic Equations.

Example 1. Find the angle x at the center of a circle subtended by a chord which cuts off a segment whose area is one-eighth of that of the circle.

Solution. If x is measured in radians and if r is the radius, the area of the segment is equal to the left member of

$$\tfrac{1}{2}r^2(x - \sin x) = \tfrac{1}{8}\pi r^2,$$

whence

$$x - \sin x = \tfrac{1}{4}\pi.$$

By means of a graph of $y = \sin x$ (which is an arch if $0 \leqq x \leqq \pi$) and the straight line represented by $y = x - \frac{1}{4}\pi$, we see that the abscissa of their point of intersection is approximately 1.78 radians or 102°. Thus $g = 102°$ is a first approximation to the root of

$$f(x) \equiv x - \sin x - \tfrac{1}{4}\pi = 0.$$

We assume from calculus that the derivative of sin x is cos x. Thus a new approximation is $g+h$, where

$$h=\frac{\quad}{f'(g)} \qquad 1-\cos g \qquad \frac{\pi}{\ }.$$

$$\begin{array}{rl} \sin 102^\circ = & 0.9781 \\ \frac{1}{4}(3.1416) = & 0.7854 \\ \hline & 1.7635 \end{array} \qquad \begin{array}{rl} \cos 102^\circ = & -0.2079 \\ 1-\cos 102^\circ = & 1.2079 \end{array}$$

$$\begin{array}{rl} 102^\circ = & 1.7802 \text{ radians} \\ \hline & -0.0167 \end{array} \qquad =\frac{-0.0167}{1.2079}=-0.0138$$

$$g_1=g+h=1.7664$$

$$h_1=\frac{-f(g_1)}{f'(g_1)}=\frac{-1.7664+0.9809+0.7854}{1.1944}=-0.0001.$$

Hence $x=g_1+h_1=1.7663$ radians, or $101^\circ\,12'$.

EXAMPLE 2. Solve $2x-\log x=7$, the logarithm being to base 10.

Solution. A table of common logarithms shows at once that a fair approximation to x is $g=3.8$. Write

$$7, \qquad \log x=M\log_e x,$$

By calculus, the derivative of $\log_e x$ is $1/x$. Hence

$$f'(x)=2-\frac{M}{x}, \qquad f'(g)=2-0.1143=1.8857,$$

$$f(g)=0.6-\log 3.8=0.6-0.57978=0.02022,$$

$$-h=\frac{f(g)}{f'(g)}=0.0107, \qquad g_1=g+h=3.7893,$$

$$f(g_1)=0.000041, \qquad f(3.7892)=-0.000148.$$

$$\frac{148}{189}\times 0.0001=0.000078, \qquad x=3.789278.$$

All figures of x are correct as shown by Vega's table of logarithms to 10 places.

PROBLEMS

Find the angle x at the center of a circle subtended by a chord which cuts off a segment whose ratio to the circle is

1. $\frac{1}{6}$. **2.** $\frac{1}{4}$, *Ans.* $132^\circ\,20.7'$. **3.** $\frac{3}{8}$, *Ans.* $157^\circ\,12'$.

When the logarithms are to base 10,

4. Solve $2x - \log x = 10$. **5.** Solve $3x - \log x = 9$, *Ans.* 3.1668771.

6. Find the angle just $>15°$ for which $\frac{1}{2} \sin x + \sin 2x = 0.64$, *Ans.* $15° 16.5'$.

7. Find the angle just $>72°$ for which $x - \frac{1}{2} \sin x = \frac{1}{4}\pi$, *Ans.* $72° 17'$.

8. Find the other solutions of Problem 6 by replacing $\sin 2x$ by $2 \sin x \cos x$, squaring, and solving the quartic equation for $\cos x$, *Ans.* $85° 56\frac{1}{2}'$, $212° 49'$, $225° 57'$.

9. Solve $\sin x + \frac{1}{2} \sin 2x = 0.7$.

10. Solve $\sin x + \sin 2x = 1.2$, *Ans.* $5° 56\frac{1}{2}'$, $25° 18'$.

11. Find x to six decimal places in $\sin x = x - 2$, *Ans.* 2.5541949.

12. Find x to five decimal places in $x = 4 \log_e x$.

13. Find x to five decimal places in $x = 3 \log_e x$, *Ans.* 1.85718, 4.53640.

14. Solve $x - \log_{10} x = 7$. Here Newton's method would be longer than the following. By glancing at a table of common logarithms, we find numbers between 7 and 8 whose logarithms coincide approximately with the decimal part of x:

$$x = 7.897, \qquad \log x = 0.89746, \qquad x - \log x = 6.99954,$$
$$7.898, \qquad 0.89752, \qquad 7.00048.$$

In the final column the ratio for interpolation is $\frac{46}{94}$, so that the correction to the upper x is $.001 \times \frac{46}{94} = .00049$. Hence $x = 7.89749$. Find a second answer $x = 1/y$, where $\log y$ is just less than 7.

15. Solve $x - \log x = 8$ by this method of interpolation.

16. What arc of a circle is double its chord? *Ans.* 3.790988. Hint: If A is the angle at the center, measured in radians, the length of the arc is A, and half the chord is of length $\sin \frac{1}{2}A$.

17. What arc of a circle is the product of its chord by $\frac{3}{2}$?

18. What arc of a circle is double the distance from the center of the circle to the chord of the arc? *Ans.* $84° 41' 34\frac{1}{2}''$.

19. If A and B are the points of contact of two tangents to a circle of radius unity from a point P without it, and if arc AB is equal to PA, find the length of the arc. *Ans.* $133° 33.8'$.

20. Find the angle at the center of a circle of a sector which is bisected by its chord. *Ans.* $108° 36' 14''$.

21. Find the radius of the smallest hollow iron sphere, with air exhausted, which will float in water if its shell is 1 inch thick and the specific gravity of iron is 7.5. *Ans.* 21.47.

22. From one end of a diameter of a circle draw a chord which bisects the semicircle. *Ans.* Angle at center is $47° 39' 13''$.

23. From one end of a diameter draw a chord which trisects the semicircle.

24. The equation $x \tan x = c$ occurs in the theory of vibrating strings. Its approximate solutions may be found from the graphs of $y = \cot x$, $y = x/c$. Find x when $c = 1$. *Ans.* $49° 17' 36.5''$.

25. The equation $\tan x = x$ occurs in the study of the vibrations of air in a spherical cavity. From an approximate solution $x_1 = 1.5\pi$, we obtain successively better approxi-

mations $x_2 = \tan^{-1} x_1 = 1.4334\pi$, $x_3 = \tan^{-1} x_2, \cdots$. Find the first three solutions to four decimal places. *Ans.* 1.4303π, 2.4590π, 3.4709π.

26. Find to three decimal places the first five solutions of

$$\tan x = \frac{2x}{2-x^2},$$

which occurs in the theory of vibrations in a conical pipe.

Ans. $x/\pi = 0.6625$, 1.891, 2.930, 3.948, 4.959.

27. Solve $x^x = 90$.

28. Solve $x = 8 \log x$.

29. Solve $x^x = 100$. *Ans.* 3.597285.

30. Solve $x = 10 \log x$. *Ans.* 10, 1.371288.

31. Solve $x + \log x = x \log x$. *Ans.* 0.326878, 12.267305.

32. Solve $x + 2 \log x = x \log x$.

33. Solve Kepler's equation $M = x - e \sin x$ when $M = 332° \, 28' \, 54.8''$, $e = 14° \, 3' \, 20''$. *Ans.* $324° \, 16' \, 29.55''$.

34. In what time would a sum of money at 6% interest compounded annually amount to as much as the same sum at simple interest at 8%. *Ans.* 10 years, 4 months, 0 days.

35. Solve Problem 34 if simple interest is at $7\frac{1}{2}\%$.

CHAPTER IX

Determinants; Systems of Linear Equations

65. Solution of Two Linear Equations by Determinants. Assume that there is a pair of numbers x and y for which

$$(1)\qquad \begin{cases} ax+by=k, \\ cx+dy=l. \end{cases}$$

Multiply the terms of the first equation by d and those of the second equation by $-b$, and add the resulting equations. We get

$$(2)\qquad (ad-bc)x=kd-lb.$$

Employing the new multipliers $-c$ and a similarly, we get

$$(3)\qquad (ad-bc)y=la-kc.$$

The common multiplier of x and y in (2) and (3) is

$$(4)\qquad D=ad-bc.$$

If $D\neq 0$, we obtain x and y by dividing the members of (2) and (3) by D

$$(5)\qquad \begin{array}{cc} kd-lb & la-kc \\ & D \end{array}$$

These values actually satisfy the proposed equations (1) since they imply

$$ax+by=\frac{(ad-bc)k}{D}=k, \qquad cx+dy=\frac{(ad-bc)l}{D}=l.$$

This proves that equations (1) have a unique set of solutions x and y if $D\neq 0$. The more troublesome case $D=0$ is postponed to §§ 82–85.

When we later treat n linear equations in n unknowns, we shall find that, if $n>2$, the expression (4) is replaced by a complicated polynomial involving many letters. We shall then need a convenient notation or symbol for such a polynomial. Even in the case of the simple expression (4), a symbol is desirable since it enables us to express also the numerators of (5) by means of the same symbol in new letters.

We recall that a point with the coordinates x and y is denoted by (x, y). Similarly, in choosing a symbol for the expression (4), we must exhibit all the letters a, b, c, d involved. It is desirable that they shall retain the same relative positions

$$\begin{matrix} c & d \end{matrix}$$

as in our equations (1). But if we enclose this array within parentheses, we obtain a symbol used later with a different meaning. It is customary to use vertical bars. Accordingly we shall employ the symbol

$$(6) \qquad \begin{matrix} a & b \\ c & d \end{matrix}$$

to denote the expression (4), which is called a *determinant* of order 2. It is also called the *determinant of the coefficients* of x and y in equations (1).

Hence relations (2) and (3) may be written in the form

$$(7) \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} x = \begin{vmatrix} k & b \\ l & d \end{vmatrix}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} y = \begin{vmatrix} a & k \\ c & l \end{vmatrix}$$

We shall call k and l the *known terms* of equations (1). Hence we have proved the following result for two linear equations in two unknowns.

Theorem 1. *If* D *is the determinant of the coefficients of the unknowns, the product of* D *by any one of the unknowns is equal to the determinant whose symbol is obtained from that for* D *by substituting the known terms in place of the coefficients of that unknown.*

We shall later find that this theorem holds for n linear equations in n unknowns.

Example. For $2x-3y=-4$, $6x-2y=2$, we have

$$\begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix} x = \begin{vmatrix} -4 & -3 \\ 2 & -2 \end{vmatrix} \qquad 14x=14, \qquad x=1,$$

$$14y = \begin{vmatrix} 2 & -4 \\ 6 & 2 \end{vmatrix} = 28, \qquad y=2.$$

PROBLEMS

Solve by determinants the following systems of equations:

1. $8x+y=34,$

2. $3x+4y=10,$ *Ans.*
   $4x+\ y=\ 9.$ $\quad y=1.$

3. $ax-by=a^2.$
   $bx+ay=ab.$

4. $x\cos A-y\sin A=0,$ *Ans.* $x=\sin A,$
   $x\sin A+y\cos A=1,$ $\quad y=\cos A.$

5. $\dfrac{5}{z}-\dfrac{2}{w}=4,$
   $\dfrac{7}{z}-\dfrac{3}{w}=5,$

6. $\dfrac{5}{z}+\dfrac{6}{w}=39,$ *Ans.* $z=\dfrac{1}{3},$
   $\dfrac{4}{z}+\dfrac{5}{w}=32,$ $\quad w=\dfrac{1}{4}.$

7. Apply Theorem 1 to $7x-5y=m,\quad 21x-15y=n.$

66. Solution of Three Linear Equations by Determinants. Consider a system of three linear equations

$$\begin{aligned} a_1x+b_1y+c_1z&=k_1,\\ a_2x+b_2y+c_2z&=k_2,\\ a_3x+b_3y+c_3z&=k_3. \end{aligned} \tag{8}$$

Multiply the members of the first, second and third equations by

$$b_2c_3-b_3c_2, \qquad b_3c_1-b_1c_3, \qquad b_1c_2-b_2c_1, \tag{9}$$

respectively, and add the resulting equations. We obtain an equation in which the coefficients of y and z are found to be zero, while the coefficient of x is

$$-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2- \tag{10}$$

which is the sum of the products of the numbers (9) by a_1, a_2, a_3, respectively. Such an expression (10) is called a *determinant of the third order* and is denoted by the symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{11}$$

The nine numbers $a_1, \cdots, c_3$ are called the *elements* of the determinant. In the symbol these elements lie in three (horizontal) *rows,* and

also in three (vertical) *columns.* Thus a_2, b_2, c_2 are the elements of the second row, while the three c's are the elements of the third column.

The equation (free of y and z), obtained above, may now be written as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$$

since the right member was the sum of the products of the expressions (9) by k_1, k_2, k_3, and hence may be derived from (10) by replacing the a's by the k's. Thus the theorem of § 65 holds here as regards the unknown x. We shall later prove, without the laborious computations just employed, that the theorem holds for all three unknowns.

67. The Signs of the Terms of a Determinant of Order 3. In the six terms of our determinant (10), the letters a, b, c were always written in this sequence, while the subscripts are the six possible arrangements of the numbers 1, 2, 3. The first term $a_1b_2c_3$ shall be called the *diagonal term,* since it is the product of the elements in the main diagonal running from the upper left-hand corner to the lower right-hand corner of the symbol (11) for the determinant. The subscripts in the term $-a_1b_3c_2$ are derived from those of the diagonal term by interchanging 2 and 3, and the minus sign is to be associated with the fact that an odd number (here one) of interchanges of subscripts were used. To obtain the arrangement 2, 3, 1 of the subscripts in the term $+a_2b_3c_1$ from the natural order 1, 2, 3 (in the diagonal term), we may first interchange 1 and 2, obtaining the arrangement 2, 1, 3, and then interchange 1 and 3; an even number (two) of interchanges of subscripts were used and the sign of the term is plus.

While the arrangement 1, 3, 2 was obtained from 1, 2, 3 by one interchange (2, 3), we may obtain it by applying in succession the three interchanges (1, 2), (1, 3), (1, 2), and in many new ways. To show that the number of interchanges which will produce the final arrangement 1, 3, 2 is odd in every case, note that each of the three possible interchanges, viz., (1, 2), (1, 3), and (2, 3), merely changes the sign of the product

(12) $$P=(x_1$$

where the x's are arbitrary variables. Thus *a succession of* k *interchanges yields* P *or* $-$P *according as* k *is even or odd.* Starting with the arrangement 1, 2, 3 and applying k successive interchanges, suppose that we obtain the final arrangement 1, 3, 2. But if in P we replace the subscripts 1, 2, 3 by 1, 3, 2, respectively, i.e., if we interchange 2 and 3, we obtain $-P$. The statement in italics shows that k is odd. We have therefore proved the following rule of signs:

Although the arrangement r, s, t *of the subscripts in any term* $\pm a_r b_s c_t$ *of the determinant may be obtained from the arrangement* 1, 2, 3 *by various successions of interchanges, the number of these interchanges is either always an even number and then the sign of the term is plus, or always an odd number and then the sign of the term is minus.*

PROBLEMS

1. Apply the rule of signs to the last three terms of (10); also to the determinant

2. If $c_1=c_2=0$, determinant (10) becomes $(a_1b_2-a$

3. The conclusion in Problem 2 holds also if $a_3=b_3=0$.

Using Problems 2 and 3, compute

4. $$\begin{matrix} 4 & 3 & 0 \\ -2 & 2 & 0 \\ a & b & 3 \end{matrix}$$

5. $$\begin{vmatrix} 4 & 3 & r \\ -2 & 2 & s \\ 0 & 0 & 3 \end{vmatrix}$$ *Ans.* 42.

68. Number of Interchanges Always Even or Always Odd. We now extend the result in § 67 to the case of n variables $x_1, \cdots, x_n$. Consider the product P of all their differences x_i-x_j $(i<j)$. We have (12) if $n=3$.

For example, let $n=4$. To find what happens to P when we interchange the subscripts 1 and 3, note that P is the product of x_2-x_4, which remains unaltered; x_1-x_3, which is changed in sign; $(x_1-x_4)(x_3-x_4)$ and $-(x_1-x_2)(x_3-x_2)$, each of which remains unaltered since its two factors are evidently merely interchanged when the subscripts 1 and 3 are interchanged. Hence if the subscripts 1 and 3 are interchanged in P, the new product is equal to $-P$.

The argument in our example may be extended to any n. Interchange any two subscripts i and j. The factors which involve neither i nor j are

unaltered. The factor $x_i - x_j$ involving both is changed in sign. The remaining factors may be paired to form the products

$$\cdot - x_k) \qquad (k=1, \cdots, n; \quad k \neq \cdot$$

Such a product is unaltered. Hence if the subscripts i and j are interchanged in P, the new product is equal to $-P$.

Suppose that an arrangement $i_1, i_2, \cdots, i_n$ can be obtained from $1, 2, \cdots, n$ by using m successive interchanges and also by t successive interchanges. Make these interchanges on the subscripts in P; the resulting functions are equal to $(-1)^m P$ and $(-1)^t P$, respectively. But the resulting functions are identical since either can be obtained at one step from P by replacing the subscript 1 by i_1, 2 by $i_2, \cdots, n$ by i_n. Hence

$$(-1)^m P \equiv (-1)^t P,$$

so that m and t are both even or both odd.

THEOREM 2. *If the same arrangement is derived from* $1, 2, \cdots, n$ *by* m *successive interchanges as by* t *successive interchanges, then* m *and* t *are both even or both odd.*

69. Definition of a Determinant of Order n. We define a determinant of order 4 to be

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \text{sum of 24 terms of type } \pm a_q b_r c_s d_t, \tag{13}$$

where q, r, s, t is any one of the 24 arrangements of 1, 2, 3, 4, and the sign of the corresponding term is $+$ or $-$ according as an even or odd number of interchanges are needed to derive this arrangement q, r, s, t from 1, 2, 3, 4. Although different numbers of interchanges will produce the same arrangement q, r, s, t from 1, 2, 3, 4, these numbers are all even or all odd, as just proved, so that the sign is fully determined.

We have seen that the analogous definitions of determinants of orders 2 and 3 lead to our earlier expressions $a_1b_2 - a_2b_1$ and (10).

We shall have no difficulty in extending the definition to a determinant of general order n as soon as we decide upon a proper notation for the n^2

elements. The subscripts $1, 2, \cdots, n$ may be used as before to specify the rows. But the alphabet does not contain n letters with which to specify the columns. The use of $a, b, \cdots, k, l$ to denote n letters would make our later proofs obscure, not to mention that l is actually the twelfth letter of the alphabet. The use of $e', e'', \cdots, e^{(n)}$ would conflict with a notation for derivatives and would be very awkward when exponents also occur.

It is now customary to denote n letters (or numbers) by $e_1, e_2, \cdots, e_n$ (or by some letter other than e with the same subscripts). To obtain the elements of the i-th row in the symbol of a determinant, we must attach the subscript i to each of $e_1, \cdots, e_n$. It is customary to place i before* $1, \cdots, n$ and hence obtain $e_{i1}, \cdots, e_{in}$. Thus the symbol of a determinant of order n is

$$D=\begin{vmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{vmatrix}, \tag{14}$$

in which the first subscript specifies the row and the second subscript fixes the column. We define (14) to be the sum of the $n!$ terms

$$(-1)^i e_{i_1 1} e_{i_2 2} \cdots e_{i_n n} \tag{15}$$

in which $i_1, i_2, \cdots, i_n$ is an arrangement of $1, 2, \cdots, n$, derived from the latter by i successive interchanges.

PROBLEMS

1. Find the six terms involving d_4 in the determinant (13) and verify that their sum is the product of d_4 by the determinant (10).

2. Show that the arrangement 4, 1, 3, 2 may be obtained from 1, 2, 3, 4 by using the two successive interchanges (1, 4), (1, 2), and also by using the four successive interchanges (1, 4), (1, 3), (1, 2), (2, 3).

3. In (14) take $n=4$ and write a_j, b_j, c_j, d_j for $e_{j1}, e_{j2}, e_{j3}, e_{j4}$, respectively. Show that we obtain (13) and that the general term (15) becomes the general term $(-1)^i a_{i_1} b_{i_2} c_{i_3} d_{i_4}$ of the second member of (13).

4. What are the signs of $a_3b_5c_2d_1e_4$ and $a_5b_4c_3d_2e_1$ in a determinant of order 5? *Ans.* $+$, $+$;

* If i is placed after $1, \cdots, n$, we obtain an equal determinant (§72).

70. Interchange of Two Rows.

THEOREM 3. *A determinant* D *is merely changed in sign by the interchange of any two rows in its symbol.*

For example,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} \qquad c-ad=-D.$$

Proof. Let Δ be the determinant which is obtained from (14) by interchanging its rth and sth rows. The terms of Δ are therefore obtained from the terms (15) of D by interchanging r and s in the series of first subscripts. In (15) the arrangement $i_1, \cdots, i_n$ (of first subscripts) is derived from the arrangement $1, \cdots, n$ by i successive interchanges. But we made one more interchange (of first subscripts) to get a term of Δ from (15). Hence the sign of this term of Δ is that $(-1)^{i+1}$. Thus $\Delta=-D$.

71. Two Rows Alike.

THEOREM 4. *A determinant is zero if any two rows of its symbol are alike.*

Proof. By the interchange of the two like rows, the determinant is evidently unaltered, and yet must change in sign by Theorem 3. Hence

EXAMPLE. Show that $D\equiv P$ if

$$\begin{matrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{matrix} \qquad P=(b-a)(c-a)(c-b).$$

Solution. By Theorem 4, $D=0$ if $a=b$. Hence by the factor theorem, $b-a$ is a factor of D. Similarly, $c-a$ and $c-b$ are factors of D. Since these factors are distinct, D has the factor P. But the terms bc^2, etc., of D are all of the third degree in a, b, c. Thus D/P is a constant. The latter is unity since bc^2 is also a term of P.

PROBLEMS

1. Prove that $\begin{vmatrix} a & b & c \\ d & e & f \\ k & l & m \end{vmatrix} = \begin{vmatrix} d & e & f \\ k & l & m \\ a & b & c \end{vmatrix}$

Find the factors of

2. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix}$.

3. $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$.

4. $\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$, *Ans.* Product of all differences $x_i - x_j$ having $i>j$.

5. $\begin{vmatrix} 1 & ab+cd & a^2b^2+c^2d^2 \\ 1 & ac+bd & a^2c^2+b^2d^2 \\ 1 & ad+bc & a^2d^2+b^2c^2 \end{vmatrix} = -(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$

6. Prove that the equation of the straight line determined by the two distinct points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

72. Interchange of Rows and Columns. To the determinant D in (14) corresponds a new determinant

$$D' = \begin{vmatrix} e_{11} & e_{21} & \cdots & e_{n1} \\ e_{12} & e_{22} & \cdots & e_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ e_{1n} & e_{2n} & \cdots & e_{nn} \end{vmatrix},$$

whose* first column is the first row of D, whose second column is the second row of D, $\cdots$, and whose n-th column is the n-th row of D. We shall say that the symbol of D' has been formed from the symbol of D by interchanging the rows and columns, or briefly that the rows and columns of D have been interchanged.

THEOREM 5. *Any determinant is not altered in value if in its symbol we interchange the rows and columns.*

* The elements of the first column of D' are the elements, taken in the same order, of the first row of D.

For example,

$$\begin{matrix} a & b \\ c & d \end{matrix} \qquad\qquad \begin{matrix} a & c \\ b & d \end{matrix}$$

Proof that $D'=D$. Define a_{ki} to be e_{ik} for all values $\leqq n$ of the subscripts. Then D' becomes

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \\ a_{n1} & a_{n2} & \cdots & a_{n} \end{matrix}$$

Since the subscripts are the same as in (14), this determinant is the sum of $n!$ terms

$$(-1)^i\, a_{i_1 1}\, a_{i_2 2} \cdots a_{i_n n},$$

in which

(16) $\begin{cases}\text{the arrangement } i_1, \cdots, i_n \text{ of } 1, \cdots, n \text{ is derived from the latter} \\ \text{by } i \text{ successive interchanges } I_1, I_2, \cdots, I_i.\end{cases}$

Replacing a_{ki} by its value e_{ik}, we conclude that D' is the sum of the $n!$ terms

(17) $$T=(-1)^i e_{1i_1}\, e_{2i_2} \cdots e_{ni_n}$$

having the same property (16). Now that we know the value of D', we are ready to prove that $D'=D$.

We shall first prove that property (16) implies

(18) $\begin{cases}\text{the arrangement } 1, \cdots, n \text{ of } i_1, \cdots, i_n \text{ is derived from the latter} \\ \text{by the successive interchanges } I_i, \cdots, I_2, I_1.\end{cases}$

To give an example, which also illustrates the proof, note that the arrangement 3, 2, 1 is derived from 1, 2, 3 by the interchange (1, 3), while the arrangement 3, 1, 2 is derived from 3, 2, 1 by (1, 2), so that the arrangement 3, 1, 2 is derived from 1, 2, 3 by the successive interchanges (1, 3) and (1, 2),—a case of (16). The same facts show that the arrangement 3, 2, 1 is derived from 3, 1, 2 by (1, 2), and the arrangement 1, 2, 3 is derived from 3, 2, 1 by (1, 3), so that the arrangement 1, 2, 3 is derived from 3, 1, 2 by the successive interchanges (1, 2) and (1, 3),—the corresponding case of (18).

Let A_2 denote the arrangement which is derived from $1, 2, \cdots, n$ (an arrangement denoted by A_1) by the interchange I_1; let A_3 denote the arrangement which is derived from A_2 by the interchange I_2, etc.; finally let A_{i+1} be derived from A_i by the interchange I_i. Hence A_{i+1} is derived from A_1 by the successive interchanges $I_1, I_2, \cdots, I_i$. By (16), A_{i+1} is

therefore the arrangement $i_1, i_2, \cdots\ i_n$. Then, conversely, A_i is derived from A_{i+1} by the interchange $I_i, \cdots, A_2$ from A_3 by I_2, and A_1 from A_2 by I_1, so that A_1 is derived from A_{i+1} by the successive interchanges $I_i, \cdots, I_2, I_1$, as stated in (18).

Without disturbing the sign, rearrange the factors of T in (17) so that in the resulting product R the second subscripts are $1, 2, \cdots, n$ in this order. Since property (16) implies (18), the arrangement $1, \cdots, n$ of the second subscripts in R can be derived from the arrangement $i_1, \cdots, i_n$ of the second subscripts in T by i successive interchanges. Since a second subscript uniquely determines a factor, the preceding sentence implies that the arrangement of the factors in R can be derived from that in T by i successive interchanges of factors. We now watch the effect on the first subscripts. We see that the arrangement of first subscripts, denoted by $k_1, \cdots, k_n$, in R can be derived from the arrangement $1, \cdots, n$ of first subscripts in T, given by (17), by i successive interchanges. Thus

$$R = (-1 \qquad\qquad e_{k_n n}$$

has the proper sign to make it a term (15) of the determinant D in (14). Evidently $T = R$. Since we have now proved that each of the $n!$ terms T of D' is equal to one of the $n!$ terms of D, we conclude that $D' = D$.

73. Interchange of Two Columns.

THEOREM 6. *A determinant* D *is merely changed in sign by the interchange of any two columns in its symbol.**

Proof. Let d denote the determinant which is derived from D by interchanging the rth and sth columns. By interchanging the rows and columns in D and in d, we get two determinants D' and d', either of which can be derived from the other by the interchange of the rth and sth rows. Hence $D' = -d'$ by Theorem 3. But $D = D'$ and $d = d'$ by Theorem 5. Hence

COROLLARY. *A determinant is zero if any two of its columns are alike.*

74. Minors. The determinant of order $n-1$ obtained by erasing (or covering up) the row and column crossing at a given element of a determinant of order n is called the *minor* of that element.

* Henceforth we shall drop the words "in its symbol."

For example, in the determinant

(19) $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

the minors of b_1, b_2, b_3 are respectively

$$B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \quad B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Again (11) is the minor of d_4 in the determinant of order 4 given by (13).

75. Expansion by a Row or Column. In the determinant (19), denote the minor of any element by the corresponding capital letter, so that b_1 has the minor B_1, b_3 has the minor B_3, etc., as in § 74. We shall prove that

$$D = - \qquad -c_2C_2,$$

$$D = \qquad\qquad D = \quad c_1C_1 - c_2C_2 + c_3C_3.$$

The three relations at the left (or right) are expressed in words by saying that *a determinant* D *of the third order may be expanded by the first, second, or third row* (*or column*). To obtain the expansion, we multiply each element of the row (or column) by the minor of the element, prefix the proper sign to the product, and add the signed products. The signs are alternately + and −, as in the diagram

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

For example, expansion by the second column gives

$$\begin{vmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \\ 3 & 0 & 9 \end{vmatrix} = -4 \begin{vmatrix} 2 & 3 \\ 3 & 9 \end{vmatrix} = -4 \times 9 = -36.$$

Similarly the value of the determinant (13) of order 4 may be found by expansion by the fourth column:

$$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} + d_2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} - d_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix} + d_4 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

THEOREM 7. *determinant* D *of order* n *may be expanded by any row or any column*

Proof. Let E_{ij} denote the minor of e_{ij} in D, given by (14), so that E_{ij} is obtained by erasing the ith row and jth column of D.

(i) We shall first prove that

$$D = e_{11} E_{11} - e_{21} E_{21} + e_{31} E_{31} - \cdots + (-1)^{n-1} e_{n1} E_{n1}, \tag{20}$$

so that D may be expanded by its first column. By (15) the terms of D having the factor e_{11} are of the form

where $1, i_2, \cdots, i_n$ is an arrangement of $1, 2, \cdots, n$, obtained from the latter by i interchanges, so that $i_2, \cdots, i_n$ is an arrangement of $2, \cdots, n$, derived from the latter by i interchanges. After removing from each term the common factor e_{11} and adding the quotients, we obtain a sum which, by definition, is the value of the determinant E_{11} of order $n-1$. Hence the terms of D having the factor e_{11} may all be combined into $e_{11} E_{11}$, which is the first part of (20).

We shall next prove that the terms of D having the factor e_{21} may be combined into $-e_{21}E_{21}$, which is the second part of (20). For, if Δ be the determinant obtained from D by interchanging its first and second rows, the result just proved shows that the terms of Δ having the factor e_{21} may be combined into the product of e_{21} by the minor

$$\begin{matrix} e_{12} & e_{13} & \cdots & e_1 \\ e_{32} & e_{33} & \cdots & e_3 \\ & & & e_{nn} \end{matrix}$$

of e_{21} in Δ. Now this minor is identical with the minor E_{21} of e_{21} in D. But $\Delta = -D$ (§ 70). Hence the terms of D having the factor e_{21} may be combined into $-e_{21}E_{21}$. Similarly, the terms of D having the factor e_{31} may be combined into $e_{31}E_{31}$, etc., as in (20).

(ii) We shall next prove that D may be expanded by its kth column as follows

$$(21)\quad D = \sum_{j=1}^{n} (-1)^{j+k} e_{jk} E_{jk} \equiv (-1)^{1+k} e_{1k} E_{1k} + \cdots + (-1)^{n+k} e_{nk} E_{nk}.$$

Consider the determinant δ derived from D by moving the kth column over the earlier columns until it becomes the new first column. Since this may be done by $k-1$ interchanges of adjacent columns, $\delta = (-1)^{k-1}D$. The minors of the elements $e_{1k}, \cdots, e_{nk}$ in the first column of δ are evidently the minors $E_{1k}, \cdots, E_{nk}$ of $e_{1k}, \cdots, e_{nk}$ in D. Hence, by (20),

$$\delta = e_{1k}E_{1k} - e_{2k}E_{2k} + \cdots + (-1)^{n-1} e_{nk} E_{nk} = \sum_{j=1}^{n} (-1)^{j+1} e_{jk} E_{jk}.$$

Thus $D = (-1)^{k-1}\delta$ has the desired value (21)

(iii) Finally, D may be expanded by its kth row:

$$D = \sum_{j=1}^{n} (-1)^{j+k} e_{kj} E_{kj}.$$

In fact, by case (ii), the latter is the expansion of the equal determinant D' in § 72 by its kth column.

76. Removal of Factors.

THEOREM 8. *A common factor of all the elements of the same row or same column of a determinant may be divided out of the elements and placed as a factor before the new determinant.*

In other words, if all of the elements of a row or column are divided by m, the value of the determinant is divided by m. For example,

$$\begin{vmatrix} ma_1 & mb_1 \\ a_2 & b_2 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad \begin{vmatrix} a_1 & mb_1 & c_1 \\ a_2 & mb_2 & c_2 \\ a_3 & mb_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Proof. Expand the determinants by the row or column in question and note that the minors are the same for the two determinants. Thus the second equation is equivalent to

$$-(mb_1)B_1+(mb_2)B_2-(mb_3)B_3=m(-b_1B_1+b_2B_2-b_3B_3),$$

where B_i denotes the minor of b_i in the final determinant.

PROBLEMS

1. $\begin{vmatrix} 3a & 3b & 3c \\ 5a & 5b & 5c \\ d & e & f \end{vmatrix}=0.$

2. $\begin{vmatrix} 2r & l & 3r \\ 2s & m & 3s \\ 2t & n & 3t \end{vmatrix}=0.$

3. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & c_2 & b_2 \\ a_1 & c_1 & b_1 \\ a_3 & c_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_3 & a_1 & a_2 \\ b_3 & b_1 & b_2 \\ c_3 & c_1 & c_2 \end{vmatrix}.$

Expand by the shortest method and evaluate

4. $\begin{vmatrix} 2 & 7 & 3 \\ 5 & 9 & 8 \\ 0 & 3 & 0 \end{vmatrix}.$

5. $\begin{vmatrix} 5 & 7 & 0 \\ 6 & 8 & 0 \\ 3 & 9 & 4 \end{vmatrix}.$

6. $\begin{vmatrix} a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = abcd(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$

7. Without computation prove that a skew-symmetric determinant of odd order is zero:

$$\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}=0. \qquad \begin{vmatrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ -b & -e & 0 & h & j \\ -c & -f & -h & 0 & k \\ -d & -a & -j & -k & 0 \end{vmatrix}=0.$$

77. Sum of Determinants.

Theorem 9. *A determinant having* a_1+q_1, a_2+q_2, $\cdots$ *as the elements of a column is equal to the sum of two determinants, one having* a_1, a_2, $\cdots$ *as the elements of the corresponding column and the other determinant having* q_1, q_2, $\cdots$ *as the elements of that column, while the elements of the remaining columns of each determinant are the same as in the given determinant.*

For example,

$$\begin{vmatrix} & b_1 & c_1 \\ & b_2 & c_2 \\ a_3+q_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} q_1 & b_1 & c_1 \\ q_2 & b_2 & c_2 \\ q_3 & b_3 & c_3 \end{vmatrix}$$

To prove the theorem we have only to expand the three determinants by the column in question (the first column in the example) and note that the minors are the same for all three determinants. Hence a_1+q_1 is multiplied by the same minor that a_1 and q_1 are multiplied by separately, and similarly for a_2+q_2, etc.

The similar theorem concerning the splitting of the elements of any row into two parts is proved by expanding the three determinants by the row in question.

For example,

$$\begin{vmatrix} a+r & b+s \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} r & s \\ c & d \end{vmatrix}$$

78. Addition to Columns or Rows.

Theorem 10. *A determinant is not changed in value if we add to the elements of any column the products of the corresponding elements of another column by the same arbitrary number.*

Proof. Let a_1, a_2, $\cdots$ be the elements to which we add the products of the elements $b_1, b_2, \cdots$ by m. We apply § 77 with $q_1=mb_1$, $q_2=mb_2$, $\cdots$. Thus the modified determinant is equal to the sum of the initial determinant and a determinant having b_1, b_2, $\cdots$ in one column and mb_1, mb_2, $\cdots$ in another column. But (§ 76) the latter determinant is equal to the product

of m by a determinant with two columns alike and hence is zero (by the corollary in § 73). For example,

$$\begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

and the last determinant is zero. A similar proof yields the next result.

THEOREM 11. *A determinant is not changed in value if we add to the elements of any row the products of the corresponding elements of another row by the same arbitrary number.*

For example,

$$\begin{vmatrix} a+mc & b+md \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + m \begin{vmatrix} c & d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

But a determinant is changed if we multiply the elements of the second row by m $(m \neq 1)$ and add the products to the elements of the first row:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \begin{vmatrix} a+mc & b+md \\ mc & md \end{vmatrix} = m \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

EXAMPLE. Evaluate the first determinant below.

$$\begin{vmatrix} 1 & -2 & 1 \\ 1 & 2 & 3 \\ 6 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 8 & 3 \\ 6 & 10 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -2 & 8 & 3 \\ 3 & 10 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 8 \\ 3 & 10 \end{vmatrix} = -44.$$

Solution. First we add to the elements of the second column the products of the elements of the last column by 2. In the resulting second determinant, we add to the elements of the first column the products of the elements of the third column by -1. Finally, we expand the resulting third determinant by its first row.

PROBLEMS

By reducing to determinants of lower order evaluate

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} \qquad \textbf{2.} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \qquad \textbf{3.} \begin{vmatrix} 1 & 2 & 3 \\ 5 & 2 & 0 \\ 3 & 2 & 7 \end{vmatrix} \quad \textit{Ans. } -44.$$

4. $\begin{vmatrix} 3 & 4 & -2 & 3 \\ -6 & 1 & 1 & 1 \\ -8 & 3 & 3 & -5 \\ 4 & 4 & -1 & 2 \end{vmatrix}.$

5. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$, *Ans.* 1.

Prove that

6. $\begin{vmatrix} a & d & 3a-4d \\ b & e & 3b-4e \\ c & f & 3c-4f \end{vmatrix} = 0.$

7. $\begin{vmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

8. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega),$

where ω is an imaginary cube root of unity.

9. $\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}$ is the product of $a+b+c+d$, $a-b+c-d$, $a+bi-c-di$, $a-bi-c+di$, where $i=\sqrt{-1}$.

10. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$

11. $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx).$

Find the factors of

12. $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}.$

13. $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$

79. System of n Linear Equations in n Unknowns with $D \neq 0$. In

$$
\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = k_1, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = k_n,
\end{array} \tag{22}
$$

let D denote the determinant of the coefficients of the n unknowns:

$$
D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{23}
$$

Then

$$
Dx_1 = \begin{vmatrix} a_{11}x_1 & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}x_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n, & a_{n2} & \cdots & a_{nn} \end{vmatrix},
$$

where the second determinant was derived from the first by adding to the elements of the first column the products of the corresponding elements of the second column by x_2, etc., and finally the products of the elements of the last column by x_n. The elements of the new first column are equal to $k_1, \cdots, k_n$ by (22). In this manner, we find that

$$
Dx_1 = K_1, \qquad Dx_2 = K_2, \cdots, \qquad Dx_n = K_n, \tag{24}
$$

in which K_i is derived from D by substituting $k_1, \cdots, k_n$ for the elements $a_{1i}, \cdots, a_{ni}$ of the ith column of D, whence

$$
K_1 = \begin{vmatrix} k_1 & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ k_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \cdots, \qquad K_n = \begin{vmatrix} a_{11} & \cdots & a_{1n-1} & k_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn-1} & k_n \end{vmatrix}.
$$

If $D \neq 0$, the unique values of $x_1, \cdots, x_n$ determined by division from (24) actually satisfy equations (22). For instance, the first equation is satisfied since

$$
k_1 D - a_{11}K_1 - a_{12}K_2 - \cdots - a_{1n}K_n = \begin{vmatrix} k_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ k_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ k_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ k_n & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},
$$

as shown by expansion by the first row; and the determinant is zero, having two rows alike.

We have therefore given a complete proof of Theorem 1.

PROBLEMS

Solve by determinants the following systems of equations (reducing each determinant to one having zero as the value of every element but one in a row or column, as in the example in § 78).

1. $x+y+z=11$, $2x-6y-z=0$, $3x+4y+2z=0$, *Ans.* $x=-8$, $y=-7$, $z=26$.

2. $x+y+z=0$, $x-y-4z=0$, $x+3y+5z=0$.

3. $x+2z=30$, $2y+z=18$, $2x+3y=21$.

4. $3x-2y=7$, $3y-2z=6$, $3z-2x=-1$ *Ans.* $x=5$, $y=4$, $z=3$.

5. $x+y+z+w=1$, $x+2y+3z+4w=11$, $x+3y+6z+10w=26$, $x+4y+10z+20w=47$, *Ans.* $x=-5$, $y=3$, $z=2$, $w=1$.

6. $2x+9y-z=2$, $x+7y+z-w=2$, $5y-2z+w=-1$, $4x-3y+2z-w=5$.

7. $x+y+z=1$, a, b, c distinct.
$ax+by+cz=k$,
$a^2x+b^2y+c^2z=k^2$, *Ans.* $x=\dfrac{(k-b)(c-k)}{(a-b)(c-a)}$, y, z by permuting a, b, c cyclically.

80. Matrix, Rank of a Determinant or Matrix. The concepts matrix and rank are essential to the clear discussion of equations (22) in the new case in which the determinant D of the coefficients is zero, as well as in the problem of m linear equations in n unknowns.

The array of coefficients of the unknowns in

$$\begin{array}{c} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=k_1, \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=k_n, \end{array} \tag{25}$$

arranged as they occur in the equations, is called the *matrix* of the coefficients and is denoted by

$$A=\left(\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{array}\right) \tag{26}$$

For example, if $m=1$, $n=2$, then $A=(a_{11}, a_{12})$. This notation is like that for a point (x, y) having the coordinates x and y. This point coincides with the point (a, b) if and only if $x=a$, $y=b$.

The terms elements, rows, and columns have the same meaning for a matrix as for a determinant. In case $m=n$, the determinant D in (23) is

called the *determinant of the square matrix* (26). Note that such a square matrix was really in the background in our definition of the symbol of a determinant.

If we erase from the matrix (26) all but r rows and all but r columns, we obtain a square matrix whose determinant (of order r) is called an *r-rowed minor* of matrix A. In particular, any element is regarded as a one-rowed minor. If $m=n$, D is regarded as an n-rowed minor of A; there was no erasure of rows or columns in this case. The minor of an element of a determinant D of order n is now called also an $(n-1)$-rowed minor of D. Examples of minors of matrix (26) are

$$a_{11}, \qquad a_{12}, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}.$$

If a matrix with $m \geqq n$ has an n-rowed minor which is not zero, the matrix is said to be of *rank n*. If all its elements are zero, a matrix is said to be of *rank zero*. But if $0<r<n$ and if some r-rowed minor of matrix A is not zero, while every $(r+1)$-rowed minor is zero, then A is said to be of rank r.

By taking $m=n$ and replacing the word matrix by determinant in the last two paragraphs we obtain the definitions of an r-rowed minor and rank of a determinant.

For example, a determinant D of order 3 is of rank 3 if $D \neq 0$; of rank 2 if $D=0$, but some 2-rowed minor is not zero; of rank 1 if every 2-rowed minor is zero, but some element is not zero; of rank 0 if every element is zero. The same statements hold for a matrix having three rows and three columns and having D as its determinant. Again, the matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix},$$

are of rank 1. Finally, the rank of matrix

$$M = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ a & b & c & d \\ e & f & g & h \end{bmatrix}$$

is $\leqq 2$ since every 3-rowed minor has two rows which are alike and hence is zero. The rank of M is 2 if some 2-rowed minor is not zero. The rank of M is 1 if a, b, c, d are not all zero and e, f, g, h are proportional to them, or vice versa, since all 2-rowed minors are then zero. The same statements, with M replaced by its determinant D, give also the rank of D.

THEOREM 12. *The rank of a matrix* A *is unaltered if we add to the elements of any column the products of the corresponding elements of another column by the same number* k.

Proof. The general proof is like that to be made for

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \qquad B=\begin{bmatrix} a_1+kb_1 & b_1 & c_1 \\ a_2+kb_2 & b_2 & c_2 \\ a_3+kb_3 & b_3 & c_3 \end{bmatrix}.$$

Let r and D denote the rank and determinant of A. Let R be the rank of B.

I. Let $r=3$. Then $D\neq 0$. The determinant of B is equal to D by Theorem 10, so that $R=3$.

II. Let $r=1$. The minors of a_1, a_2, a_3 in A are the same as the minors of a_1+kb_1, etc., in B. The minors of c_1, c_2, c_3 in A and B are equal by Theorem 10. The minor of b_1 in B is

$$M=\begin{vmatrix} a_2+kb_2 & c_2 \\ a_3+kb_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+k\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

Since all 2-rowed minors of A are zero, the same is therefore true of B. But if every element of B were zero, the same would be true for A, contrary to the hypothesis $r=1$. Hence $R=1$.

III. Let $r=2$. Suppose that all 2-rowed minors of B are zero. By case II, the minors $A_1, A_2, A_3, C_1, C_2, C_3$ of $a_1, \cdots$ in A are zero. Hence $0=M=B_1+kA_1$, so that $B_1=0$. Examining similarly the minors of b_2 and b_3 in B, we see that $B_2=B_3=0$. Hence all nine 2-rowed minors of A are zero. This contradicts $r=2$. Hence B has a non-vanishing 2-rowed minor. Finally, the determinant of B is equal to that of A and hence is zero. Thus $R=2$.

IV. Let $r=0$. Then every element of A is zero. Hence $R=0$.

COROLLARY. *Theorem* 12 *holds if the word column is replaced by row.*

Example 1. Find the rank of matrix

$$\begin{matrix} 5 & 0 & -1 & 2 \\ 4 & -1 & 1 & 5 \\ 2 & -3 & 5 & 11 \\ 1 & 1 & -2 & -3 \end{matrix}$$

Solution. We first get a zero in place of the element 5 of the first row by adding to the elements of the first column the products of those in the third column by 5. Similarly we get a zero in place of the element 2 of the first row by adding to the elements of the fourth column the products of those in the third column by 2. We may then at once get zeros in place of the elements 1, 5, -2 in the third column. We now have F.

$$\begin{vmatrix} 0 & 0 & -1 & 0 \\ 9 & -1 & 0 & 7 \\ 27 & -3 & 0 & 21 \\ -9 & 1 & 0 & -7 \end{vmatrix}, \quad G=\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad H=\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To the elements of the first (or fourth) column of F add the products of those in the second column by 9 (or 7). We get G. To the elements of the third (or fourth) row of G add the products of those in the second row by -3 (or 1). We get H. The rank of H is evidently 2. But E has the same rank as H by Theorem 12 and the corollary.

The form of H suggests a shorter, but less natural solution.

Second Solution. To the elements of the fourth row of E add those of the second row and the negatives of those of the first row. To the elements of the third row add the products of those of the first row by 2 and the products of those of the second row by -3. We get a matrix whose first two rows are the same as those of E, while all elements in the last two rows are zero.

PROBLEMS

Preserve answers for use in later problems, which will be numbered consecutively with the present problems. Find the rank r of

1. $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & -1 \\ 2 & 1 & -4 \end{vmatrix}$. **2.** $\begin{bmatrix} 1 & -3 & 4 \\ 4 & -12 & 16 \\ 3 & -9 & 12 \end{bmatrix}$. **3.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 6 \\ 6 & 2 & 9 \end{bmatrix}$

4. $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}$ (*i*) if $a \neq 1$ and $a \neq -2$; (*ii*) if $a=1$; (*iii*) if $a=-2$.

Find the rank r of the matrix of the coefficients of

5. $x+$
$x+2y+2z,$
$y-\ z,$

6. $2x-\ \ y+\ 4z,$
$x+\ 3y-\ 2z,$

7. $5x\qquad -\ z,$
$4x-\ y+\ z,$
$2x-3y+5z,$
$x+\ y-2z.$

8.

$-\ z-24w.$

9. $5z-\ 4w,$
$2z-\ \ w,$

10. The rank of a matrix is unchanged if we interchange two rows or two columns, or if we interchange rows and columns, or if we multiply every element of one row by the same number not zero.

11. If the rank of a matrix A is r, every $(r+2)$-rowed minor of A is zero, every $(r+3)$-rowed minor is zero, etc.

81. One Linear Function a Linear Combination of Other Functions.

We shall call $5(x+2y)-4(2x+3y)$ a linear combination of $x+2y$ and $2x+3y$. The latter linear functions are called *homogeneous* since they lack constant terms. But $x+2y+5$ is not homogeneous.

Lemma 1. *Consider the following linear homogeneous functions and abbreviations* L_1, *etc., for them:*

$$L_1=a_{11}x_1+\cdots+a_{1n}x_n,\ \cdots,\ L_{r+1}=a_{r+1,1}x_1+\cdots+a_{r+1,n}x_n, \tag{27}$$

where $n\geqq r+1$. *Let the matrix* A *of their coefficients be of rank* r. *Let one of its non-vanishing* r*-rowed minors be*

$$\begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \cdot & \cdot & \cdot \\ a_{r1} & \cdots & a_{rr} \end{vmatrix} \neq 0. \tag{28}$$

Let $d_1, \cdots, d_{r+1}$ *denote the minors of* $L_1, \cdots, L_{r+1}$, *respectively, in*

$$\Delta=\begin{vmatrix} a_{11} & \cdots a_{1r} & L_1 \\ \cdot & \cdot & \cdot \\ a_{r+1,1} & \cdots a_{r+1,r} & L_{r+1} \end{vmatrix}, \tag{29}$$

so that d_{r+1} *is the determinant* (28). *Then*

$$(-1)^r d_1 L_1+(-1)^{r-1} d_2 L_2+\cdots+d_{r+1} L_{r+1}\equiv 0, \tag{30}$$

identically in the variables $x_1, \cdots, x_n$. *Hence* L_{r+1} *is identically equal to a linear combination, with constant coefficients, of* $L_1, \cdots, L_r$.

Proof. The left member of (30) is the expansion of Δ by its last column (see § 75 as to the signs). Thus it remains only to prove that Δ is identically zero. By (27), Δ is the sum of n determinants, the s-th one of which differs from Δ only in having $a_{1s}x_s, \cdots, a_{r+1,s}x_s$ as the elements of the last column, and hence is the product of x_s by

$$\begin{vmatrix} a_{11} & \cdots & a_{1r} & a_{1s} \\ \cdot & \cdot & \cdot & \cdot \\ a_{r+1,1} & \cdots & a_{r+1,r} & a_{r+1,s} \end{vmatrix}.$$

If $s \leqq r$, this determinant has two columns alike and hence is zero. If $s>r$, it is an $(r+1)$-rowed minor of matrix A of rank r and hence is zero. This proves the identity (30). Transpose all its terms except the last and divide by d_{r+1}, which is not zero. This proves the final statement in the lemma.

We shall now discard the assumption (28). However, A has at least one r-rowed minor $M \neq 0$. We can rearrange the functions (27) and relabel the variables so that M will lie in the first r rows and first r columns of the matrix of the new functions. The latter will be called the *arranged functions*.

For example, let the given functions and M be

$$a_1x+b_1y+c_1z, \qquad a_2x+b_2y+c_2z, \qquad a_3x+b_3y+c_3z, \qquad M=\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

We write X for x, Y for z, Z for y, and put the first function below the other two. We obtain the arranged functions

$$a_2X+c_2Y+b_2Z,$$
$$a_3X+c_3Y+b_3Z,$$
$$a_1X+c_1Y+b_1Z.$$

Now M is the determinant of the coefficients of X and Y in the first two arranged functions.

Hence by Lemma 1 the last new function is a linear combination of the first r new functions. After restoring the original labels for the variables,

we see that the new functions become the old ones rearranged. We have therefore generalized Lemma 1 as follows.

LEMMA 2. *Consider* $r+1$ *linear homogeneous functions of* n *variables, where* $n \geqq r+1$. *If the matrix of their coefficients is of rank* r, *one of the functions is identically equal to a linear combination of the remaining* r *functions, selected so that the matrix of their coefficients has a non-vanishing* r*-rowed minor.*

Evidently Lemma 2 implies the following result.

THEOREM 13. *If* $m>r$ *and* $n>r$, *and if the matrix of the coefficients of* m *linear homogeneous functions of* n *variables is of rank* r, *then* $m-r$ *of the functions are identically equal to linear combinations of the remaining* r *functions, selected so that the matrix of their coefficients has a non-vanishing* r*-rowed minor.*

THEOREM 14. *If the determinant of the coefficients of* n *linear homogeneous functions* L_i *of* $x_1, \cdots, x_n$ *is zero there exist constants* c_i *not all zero such that*

$$c_1L_1+\cdots+c_nL_n \equiv 0$$

identically in $x_1, \cdots, x_n$.

Proof. If r denotes the rank of the determinant, then $r<n$. By the case $m=n$ of Theorem 13, $n-r$ of the L_i are linear combinations of the remaining r functions L_j. Any one of these relations may be written as the identity in Theorem 14.

EXAMPLE. For the functions in Problem 5 find the identities described in Theorem 13.

Solution. Denote the functions by L_1, L_2, L_3. Their rank r was shown to be 2 Here (28) and (29) become

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \qquad \begin{vmatrix} 1 & 1 & L_1 \\ 1 & 2 & L_2 \\ 1 & 5 & L_3 \end{vmatrix} \equiv 0.$$

Expanding the last determinant by its third column, we obtain the answer

$$3L_1-4L_2+L_3 \equiv 0 \quad \text{or} \quad L_3 \equiv -3L_1+4L_2.$$

For certain problems we must use Lemma 2 instead of Lemma 1.

PROBLEMS

12. Find similarly the identity for Problem 6, page 128.

13. For Problems 7, 8, 9, denote the functions in order by $L_1, L_2, \cdots$, recall the values of r, and find the identities described in Theorem 13.

Ans. for Problem 7: $L_3 \equiv -2L_1+3L_2$, $L_4 \equiv L_1-L_2$.
Ans. for Problem 8: $L_4 \equiv L_3-L_1-L_2$.
Ans. for Problem 9: $L_3 \equiv 2L_1+L_2$, $L_4 \equiv 3L_1-L_2$.

Find the identities for

14. $6x-9y+12z$, $8x-12y+16z$, $x+2y-3z$.

15. $3x-2y$, $3y-2z$, $x+y+z$, $2x-3z$.

16. $2y+z$, $x+y+8z$, $x+2z$, $2x+3y$.

82. Linear Homogeneous Equations. Theorem 13 implies the following result.

THEOREM 15. *Let* m>r *and* n>r *and let the matrix of the coefficients of the equations*

$$(31)\qquad \begin{array}{c} a_{11}x_1+\cdots+a_{1n}x_n=0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{m1}x_1+\cdots+a_{mn}x_n=0 \end{array}$$

be of rank r. *Select* r *of the equations so that the matrix of their coefficients has a non-vanishing* r*-rowed minor* M. *Transpose the terms whose coefficients do not appear in* M *and solve the resulting* r *equations as in* § 79. *In this manner,* r *of the unknowns are expressed uniquely as linear homogeneous functions of the remaining* n−r *unknowns, which may be assigned arbitrary values. The expressions for these* r *unknowns satisfy all the given equations* (31) *for arbitrary values of those* n−r *unknowns.*

COROLLARY. *A necessary and sufficient condition that* n *linear homogeneous equations in* n *unknowns shall have solutions not all zero is that the determinant of the coefficients be zero.*

The condition is necessary by § 79 and sufficient by Theorem 15.

EXAMPLE. Equate to zero the three functions in Problem 5 and solve the resulting equations.

Solution. In view of the preceding example, we may discard the third equation. For the remaining equations

$$y = -2z,$$

the determinant of the coefficients is unity, whence $x=-4z$, $y=z$, while z is arbitrary. These values should (and do) satisfy the third equation identically in z.

PROBLEMS

Recalling the ranks in Problems 1–3, page 127, and the answers to Problems 14–16, solve

17. $2x+y+3z=0,\quad 4x+2y-z=0,\quad 2x+y-4z=0,$

18. $x-3y+4z=0,\quad 4x-12y+16z=0,\quad 3x-9y+12z=0.$

19. $2x+y+3z=0,\quad 4x-y+6z=0,\quad 6x+2y+9z=0.$

20. $6x-9y+12z=0,\quad 8x-12y+16z=0,\quad x+2y-3z=0.$

21. $3x-2y=0,\quad 3y-2z=0,\quad x+y+z=0,\quad 3z-2x=0.$

22. $2y+z=0,\quad x+y+8z=0,\quad x+2z=0,\quad 2x+3y=0.$

23. Using the answers to Problems 12 and 13, solve the systems of equations obtained by equating to zero the functions in Problems 6–9.

Ans. for Problem 7: $y=9x$, $z=5x$, x arbitrary.
Ans. for Problem 8: $x=6w$, $y=3w$, $z=12w$, w arbitrary.
Ans. for Problem 9: $z=-\frac{17}{3}x-\frac{10}{3}y$, $w=-\frac{19}{3}x-\frac{11}{3}y$, x and y arbitrary.

24. If the matrix A of the coefficients of three linear homogeneous equations in four unknowns has rank 3, the values of the unknowns are proportional to the four 3-rowed minors of A.

83. Augmented Matrix. We define the *augmented matrix* of equations (25) to be

$$B=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & k_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & k_m \end{bmatrix}. \tag{32}$$

It is obtained from matrix A in (26) by annexing a new column.

If r is the rank of matrix A, it has an r-rowed minor which is not zero. Since this is also a minor of B, the rank of B is $\geqq r$.

THEOREM 16. *The rank of* B *is* $\geqq$r, *but cannot exceed* r+1, *where* r *is the rank of* A.

Proof. Let the rank of B be $r+s$, where $s \geqq 2$. Then B has a non-vanishing $(r+s)$-rowed minor M. In all the columns of M, except possibly the last, the elements are a's. Expand M by its last column. The minor of such an element is of order $r+s-1 \geqq r+1$ and is a minor of matrix A, since its elements are all a's. Since A is of rank r, its minors of order $\geqq r+1$ are all zero. Hence the expansion of M is zero. Thus $M=0$. This contradiction excludes the case $s \geqq 2$.

84. Inconsistent Linear Equations. We shall call two or more equations *inconsistent* (or *consistent*) if there do not (or do) exist values of the unknowns which satisfy all the equations. For example, $2x+3y=5$ and $4x+6y=8$ are evidently inconsistent; they represent two parallel lines having no point of intersection.

THEOREM 17. *Any linear equations are inconsistent if the rank of the augmented matrix* B *exceeds the rank of the matrix* A *of their coefficients.*

Proof. Let r denote the rank of A. By the hypothesis and Theorem 16, the rank of B is exactly $r+1$. Any $(r+1)$-rowed minor of B which has a's in every column is zero since the rank of A is r. Hence B has a non-vanishing $(r+1)$-rowed minor having the k's in its last column. As in § 81 we pass to the arranged equations and write them as in (25). Hence

$$K=\begin{vmatrix} a_{11} & \cdots a_{1r} & k_1 \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ a_{r+1,1} & \cdots a_{r+1,r} & k_{r+1} \end{vmatrix} \neq 0. \tag{33}$$

Suppose that our first $r+1$ arranged equations hold for the same values $X_1, \cdots, X_n$ of $x_1, \cdots, x_n$. In the proof of Lemma 1 we showed that the determinant Δ in (29) is identically zero. But for $x_1=X_1$, etc., Δ becomes K since then $L_1=k_1$, etc. Thus $K=0$. This contradiction to (33) shows that our first $r+1$ arranged equations are inconsistent. The same is therefore true for the given equations.

EXAMPLE 1. Discuss the equations

$$ax+y+z=a-3,$$
$$x+ay+z=-2,$$
$$x+y+az=-2,$$

when the determinant D of the coefficients is zero.

Solution. We find that $D=(a-1)^2(a+2)$. If $a=1$, the equations all reduce to $x+y+z=-2$ and merely determine x in terms of y and z, which are arbitrary. Next, let $a=-2$. From the last two equations we see by subtraction that $z=y$. Then the first two equations become

which are obviously inconsistent. To obtain this result by Theorem 17, note that the rank of D (or A) is 2, while the rank of the augmented matrix is 3 since it has the 3-rowed minor

$$\begin{vmatrix} 1 & 1 & -5 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{vmatrix} = -27.$$

EXAMPLE 2. Discuss the equations obtained by equating the functions in Problem 7 to 2, 5, k, l, respectively.

Solution. By Problem 13 the equations are inconsistent unless $k=11$, $l=-3$.

PROBLEMS

Prove that the following sets of equations are inconsistent except for a certain value of k, or values of k and l.

25. $\begin{aligned} 2x+y+3z&=1,\\ 4x-y+6z&=k,\\ 6x+2y+9z&=4. \end{aligned}$

26. $\begin{aligned} 2x+y+3z&=6,\\ 4x+2y-z&=7,\\ 2x+y-4z&=k, \end{aligned}$ *Ans.* $k=1$.

27. $\begin{aligned} x+y+3z&=1,\\ x+2y+2z&=2,\\ x+5y-z&=k. \end{aligned}$

28. $\begin{aligned} 2x-y+4z&=5,\\ x+3y-2z&=2,\\ x-11y+14z&=k, \end{aligned}$ *Ans.* $k=4$.

29. $\begin{aligned} 3x-2y&=7,\\ 3y-2z&=6,\\ x+y+z&=k,\\ 2x-3z&=1. \end{aligned}$

30. $\begin{aligned} 2y+z&=18,\\ x+y+8z&=k,\\ x+2z&=30,\\ 2x+3y&=21. \end{aligned}$

31. $\begin{aligned} 6x-9y+12z&=12,\\ 8x-12y+16z&=k,\\ x+2y-3z&=4, \end{aligned}$
Ans. $k=16$.

32. Equations obtained by equating the functions in Problem 8 to 2, -6, 18, k, respectively. See Problem 13.

33. Equations obtained by equating the functions in Problem 9 to 6, 9, k, l, respectively. *Ans.* $k=21$, $l=9$.

34. Any $n+1$ linear equations in n unknowns are inconsistent if the determinant of the augmented matrix is not zero.

85. Consistent Equations.

THEOREM 18. *Any* m *linear equations in* n *unknowns are consistent if and only if the rank of the matrix of the coefficients is equal to the rank of the*

augmented matrix. If the rank of both matrices is r, *certain* r *of the equations determine uniquely* r *of the unknowns as linear homogeneous functions of the remaining* n−r *unknowns, whose values are arbitrary, and the expressions for these* r *unknowns satisfy all the proposed equations.*

To prove the second part of the theorem, we pass to the arranged equations $L_1=k_1$, etc. (§ 81), such that the determinant (28) is not zero, and such that $L_{r+1}=k_{r+1}$ is the new form of any chosen one of the proposed equations other than the first r new equations $L_1=k_1, \cdots, L_r=k_r$. We proved the identity (30). The determinant (33) is now zero since the rank of the augmented matrix is r. Its expansion by the last column is

$$(-1)^r d_1k_1+(-1)^{r-1}d_2k_2+\cdots+d_{r+1}k_{r+1}=0.$$

Subtracting this from identity (30) and writing E_i for L_i-k_i, we get

$$(-1)^r d_1E_1+\cdots+d_{r+1}E_{r+1}\equiv 0.$$

Since $d_{r+1}\neq 0$, E_{r+1} is identically equal to a linear combination of $E_1, \cdots, E_r$. After their constant terms k_i are transposed, the arranged equations become $E_1=0$, $E_2=0$, etc. We have now proved that all the new equations are identically equal to linear combinations of r of them. The same is therefore true of the proposed equations. Hence the second part of our theorem follows exactly as in Theorem 15.

We readily prove the first part of Theorem 18. If matrices A and B have the same rank r, we just proved that the equations have solutions and hence are consistent. By Theorem 16, there remains only the case in which A has rank r and B has rank $r+1$. The equations are then inconsistent by Theorem 17.

PROBLEMS

Solve the following systems of consistent equations.

35. $2x+y+3z=10,\quad 4x+2y-z=13,\quad 2x+y-4z=3.$

36. $5x-z=2,\quad 4x-y+z=5,\quad 2x-3y+5z=11,\quad x+y-2z=-3,$ *Ans.* $y=9x-7$, $z=5x-2$, x arbitrary.

37. $3x-2y=7,\quad 3y-2z=6,\quad x+y+z=12,\quad 2x-3z=1,$ *Ans.* $x=5$, $y=4$, $z=3$.

38. $2y+z=18,\quad x+y+8z=105,\quad x+2z=30,\quad 2x+3y=21.$

39. $6x-9y+12z=12,\quad 8x-12y+16z=16,\quad x+2y-3z=4,$ *Ans.* $x=(z+20)/7$, $y=(10z+4)/7$.

40. $x+\ y+\ z=1,\quad a\neq c,$ *Ans.* $z=\dfrac{a-k}{a-c},$

$ax+\ ay+\ cz=k,\quad k=a$ $y=\dfrac{k-c}{a-c}-x,$

$a^2x+a^2y+c^2z=k^2,\quad$ or $\quad k=c.$ x arbitrary.

41. Show that the equations in Problem 40 are inconsistent unless $k=a$ or $k=c$.

42. Solve the equations in Problems 25–28, 32, 33, when they are consistent.

Ans. for Problem 27: $k=5,\quad x=-4z,\quad y=z+1,\quad z$ arbitrary.

Ans. for Problem 32: $k=22,\quad x=6w+2,\quad y=3w+3,\quad z=12w-2,\quad w$ arbitrary.

Ans. for Problem 33: $x=11z-10w,\quad y=-19z+17w+3,\quad z$ and w arbitrary.

43. Find the most general linear homogeneous function of x, y, z such that the rank of the determinant of the coefficients of it and $x+2y+3z$ and $x-2z$ shall be 2.

44. Discuss the equations

$$ax+by+cz=k,$$
$$a^2x+b^2y+c^2z=k^2,$$
$$a^4x+b^4y+c^4z=k^4.$$

Ans. If a, b, c are distinct and not zero, and if $a+b+c\neq 0$, then

$$x=\frac{k(b-k)(c-k)(k+b+c)}{a(b-a)(c-a)(a+b+c)},$$

while y and z are found from the value of x by permuting a, b, c cyclically. If $b=a\neq c$ and $ac\neq 0$, the equations are inconsistent unless k has one of the values $0, a, c, -a-c$; while if k has one of these values,

$$z=\frac{k(k-a)}{c(c-a)},\qquad y=\frac{k(c-k)}{a(c-a)}-x,\qquad x \text{ arbitrary.}$$

The case $a+b+c=0$ is left to the reader.

86. Complementary Minors. The determinant

$$D=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \tag{34}$$

is said to have the *two-rowed complementary minors*

$$M=\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix},\qquad M'=\begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix},$$

since either is obtained by erasing from D all the rows and columns having an element which occurs in the other.

In general, if we erase from a determinant D of order n all but r rows and all but r columns, we obtain a determinant M of order r called an r-rowed minor of D. But if we had erased from D the r rows and r columns previously kept, we would have obtained an $(n-r)$-rowed minor of D called the *minor complementary to* M. In particular, any element is regarded as a one-rowed minor and is complementary to its minor (of order $n-1$).

87. Laplace's Development by Columns.

THEOREM 19. *Any determinant* D *is equal to the sum of all the signed products* $\pm$MM′, *where* M *is an* r*-rowed minor having its elements in the first* r *columns of* D, *and* M′ *is the minor complementary to* M, *while the sign is* $+$ *or* $-$ *according as an even or odd number of interchanges of rows of* D *will bring* M *into the position occupied by the minor* M_1 *whose elements lie in the first* r *rows and first* r *columns of* D.

For $r=1$, this development becomes the known expansion of D by the first column (§ 75); here $M_1=e_{11}$.

If $r=2$ and D is the determinant (34),

$$D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\cdot\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}-\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}\cdot\begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix}+\begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix}\cdot\begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix}$$
$$+\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\cdot\begin{vmatrix} c_1 & d_1 \\ c_4 & d_4 \end{vmatrix}-\begin{vmatrix} a_2 & b_2 \\ a_4 & b_4 \end{vmatrix}\cdot\begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix}+\begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}\cdot\begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$$

The first product in the development is M_1M_1'; the second product is $-MM'$ (in the notations of § 86), and the sign is minus since the interchange of the second and third rows of D brings this M into the position of M_1. The sign of the third product in the development is plus since two interchanges of rows of D bring the first factor into the position of M_1.

)f of Theorem 19. If D is the determinant (14), then

$$M_1=\begin{vmatrix} e_{11}\cdots e_{1r} \\ \cdot\quad\cdot\quad\cdot \\ e_{r1}\cdots e_{rr} \end{vmatrix},\qquad M_1'=\begin{vmatrix} e_{r+1,r+1}\cdots e_{r+1,n} \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ e_{n,r+1}\quad\cdots e_{nn} \end{vmatrix}.$$

By (15), any term of the product M_1M_1' is of the type

$$(-1)^i e_{i_1 1}\, e_{i_2 2}\cdots e_{i_r r}\cdot(-1)^j e_{i_{r+1},\, r+1}\cdots e_{i_n n}, \tag{35}$$

where $i_1, \cdots, i_r$ is an arrangement of $1, \cdots, r$ derived from $1, \cdots, r$ by i interchanges, while $i_{r+1}, \cdots, i_n$ is an arrangement of $r+1, \cdots, n$ derived by j interchanges. Hence $i_1, \cdots, i_n$ is an arrangement of $1, \cdots, n$ derived by $i+j$ interchanges, so that the product (35) is a term of D with the proper sign.

It now follows from § 70 that any term of any of the products $\pm MM'$ mentioned in the theorem is a term of D. Clearly we do not obtain twice in this manner the same term of D.

Conversely, any term t of D occurs in one of the products $\pm MM'$. Indeed, t contains as factors r elements from the first r columns of D, no two being in the same row, and the product of these is, except perhaps as to sign, a term of some minor M. Similarly, the product of the remaining factors of t is, apart from sign, a term of the complementary minor M'. Thus t is a term of MM' or of $-MM'$. In view of the earlier discussion, the sign of t is that of the corresponding term in $\pm MM'$, where the latter sign is given by the theorem.

88. Laplace's Development by Rows. There is a Laplace development of D in which the r-rowed minors M have their elements in the first r rows of D, instead of in the first r columns as in § 87. To prove this, we have only to apply § 87 to the equal determinant obtained by interchanging the rows and columns of D.

There are more general (but less used) Laplace developments in which the r-rowed minors M have their elements in any chosen r columns of D. It is simpler to apply the earlier developments to the determinant $\Delta = \pm D$ obtained by interchanges of columns of D such that the elements of the chosen r columns of D become the elements of the first r columns of Δ.

Similarly, when the word column is replaced by row.

PROBLEMS

1. Prove that

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & j & k \\ 0 & 0 & l & m \end{vmatrix} = \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} j & k \\ l & m \end{vmatrix}.$$

2. By employing 2-rowed minors from the first two rows, show that

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ a & b & c & d \\ e & f & g & h \end{vmatrix} = \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} c & d \\ g & h \end{vmatrix} - \begin{vmatrix} a & c \\ e & g \end{vmatrix} \cdot \begin{vmatrix} b & d \\ f & h \end{vmatrix} + \begin{vmatrix} a & d \\ e & h \end{vmatrix} \cdot \begin{vmatrix} b & c \\ f & g \end{vmatrix} = 0.$$

3. By employing 2-rowed minors from the first two columns of the 4-rowed determinant in Problem 2, show that the products in Laplace's development cancel.

89. Product of Determinants.

THEOREM 20. *The product of two determinants of the same order is equal to a determinant of like order in which the element of the* r*th row and* c*th column is the sum of the products of the elements of the* r*th row of the first determinant by the corresponding elements of the* c*th column of the second determinant.*

For example,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{vmatrix} \tag{36}$$

While for brevity we shall give the proof of Theorem 20 for determinants of order 3, the method is seen to apply to determinants of any order. By Laplace's development with $r=3$ (§ 88), we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & e_1 & f_1 & g_1 \\ 0 & -1 & 0 & e_2 & f_2 & g_2 \\ 0 & 0 & -1 & e_3 & f_3 & g_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \\ e_3 & f_3 & g_3 \end{vmatrix} \tag{37}$$

In the determinant of order 6, add to the elements of the fourth, fifth, and sixth columns the products of the elements of the first column by e_1, f_1, g_1, respectively (and hence introduce zeros in place of the former elements e_1, f_1, g_1). Next, add to the elements of the fourth, fifth, and sixth columns the products of the elements of the second column by

$e_2, f_2, g_2,$ respectively. Finally, add to the elements of the fourth, fifth, and sixth columns the products of the elements of the third column by $e_3, f_3, g_3,$ respectively. The new determinant is

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1e_1+b_1e_2+c_1e_3 & a_1f_1+b_1f_2+c_1f_3 & a_1g_1+b_1g_2+c_1g_3 \\ a_2 & b_2 & c_2 & a_2e_1+b_2e_2+c_2e_3 & a_2f_1+b_2f_2+c_2f_3 & a_2g_1+b_2g_2+c_2g_3 \\ a_3 & b_3 & c_3 & a_3e_1+b_3e_2+c_3e_3 & a_3f_1+b_3f_2+c_3f_3 & a_3g_1+b_3g_2+c_3g_3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}.$$

By Laplace's development, this is equal to the 3-rowed minor whose elements are the long sums. Hence this minor is equal to the product in the right member of (37).

PROBLEMS

1. Prove (36) by means of § 77.

2. If A_i, B_i, C_i are the minors of a_i, b_i, c_i in the determinant D defined by the second factor below, prove that

$$\begin{vmatrix} A_1 & -A_2 & A_3 \\ -B_1 & B_2 & -B_3 \\ C_1 & -C_2 & C_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix}.$$

Hence the first factor is equal to D^2 if $D \neq 0$. It is called the *adjoint* of D.

3. Evaluate the adjoint of a 4-rowed determinant.

4. Prove that

$$\begin{vmatrix} aa'+bb'+cc' & ea'+fb'+gc' \\ ae'+bf'+cg' & ee'+ff'+gg' \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} a' & b' \\ e' & f' \end{vmatrix} + \begin{vmatrix} a & c \\ e & g \end{vmatrix} \cdot \begin{vmatrix} a' & c' \\ e' & g' \end{vmatrix} + \begin{vmatrix} b & c \\ f & g \end{vmatrix} \cdot \begin{vmatrix} b' & c' \\ f' & g' \end{vmatrix}.$$

5. Express $(a^2+b^2+c^2+d^2)(e^2+f^2+g^2+h^2)$ as a sum of four squares by using $i=\sqrt{-1}$ and writing

$$\begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix} \cdot \begin{vmatrix} e+fi & g+hi \\ -g+hi & e-fi \end{vmatrix}$$

as a determinant of order 2 similar to each factor. Hint: If k' denotes the conjugate of the complex number k, each of the three determinants is of the form

$$\begin{matrix} k & l \\ -l' & k' \end{matrix}$$

90. Properties of Matrices. We shall now consider only n-rowed square matrices. We define the product of two such matrices to be the matrix whose elements are found by the rule in Theorem 20. For example, as in (36) we have

$$(38) \qquad \left.\begin{matrix} b \\ d \end{matrix}\right) \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

In a product of two determinants, we may first interchange the rows and columns of the second factor before applying the "row by column" rule in Theorem 20. This proves that we could use the "row by row" rule to find the product of the given determinants, so that the element in the first row and second column of the product (36) is now $ag+bh$. The latter is the sum of the products of the elements a, b in the first row of the first factor by the elements g, h in the second row of the second factor. Similarly, we can find a correct product of determinants by a "column by column" rule or by a "column by row" rule. There are many valid reasons why these last three rules should be avoided. In the case of matrices, they must be avoided, since only the "row by column" rule is correct.

We define the sum of two matrices to be the matrix obtained by adding corresponding elements of the given matrices. For example,

$$\left.\begin{matrix} b \\ d \end{matrix}\right) \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

We define the *scalar matrix* S_t as the matrix whose elements in the main diagonal are all equal to t, while the remaining elements are all zero. The matrix which is obtained from a matrix M by multiplying all its elements by k will be denoted by M_k. Thus if $n=2$,

$$(40) \quad M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad M_k=\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}, \qquad S_t=\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

For any n it is seen immediately that

$$(41) \qquad S_k\, M=M_k, \qquad M\, S_k=M_k, \qquad \text{whence } S_1\, M=M\, S_1=M.$$

In particular, if M is S_t, then M_k is S_{kt}. Hence

$$(42) \qquad S_k\, S_t = S_{kt}, \qquad S_k + S_t = S_{k+t}.$$

Consider the correspondence between numbers k and scalar matrices S_k. By (42), this correspondence is preserved under multiplication and addition. Hence the system of all scalar matrices has the same properties as our number system. While we cannot identify a matrix with a number, we may identify S_k with kS_1 and with S_1k, since, by (41), S_1 plays the role of unity in multiplication. Then formulas (42) reduce to the trivial relations

$$(42') \qquad kS_1 \cdot tS_1 = (kt)S_1, \qquad kS_1 + tS_1 = (k+t)S_1.$$

It is customary to write 1 for S_1 and to suppress it when it is one of the factors in a product of matrices. Then (41) becomes

(43)

For $n=2$, this is equivalent to

$$(44) \qquad k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} k \quad \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

In general, the product of a number and a matrix M in either order gives the matrix whose elements are the products of those of M by k. This property of matrices is in marked contrast to Theorem 8, which permits us to remove a common factor from the elements of a *single* row (or single column) of a *determinant* and place that factor as a multiplier before the new determinant. It was with this contrast in mind that we explained in such detail the origin of formula (43), rather than take it as a definition.

PROBLEMS

1. Multiplication of matrices is usually not commutative. If

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \text{then } MN = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad NM = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

2. For $n=2$, verify that the associative law holds: $AB \cdot C = A \cdot BC$.

3. Define the *adjoint* of a matrix M in the same manner as we defined the adjoint of a determinant in Problems 2, 3 of § 89. Hence the product of M by its adjoint in either order is the scalar matrix S_D, where D is the determinant of M.

4. If $D \neq 0$, write M^{-1} for the product of $1/D$ by the adjoint of M. Why is $M^{-1} = M^{-1}M = 1$? Then M^{-1} is called the *inverse* of M. Which matrix X solves $XM = N$? Which Y solves $MY = N$?

5. For $i = \sqrt{-1}$, consider the matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Verify that $I^2 = J^2 = K^2 = -1$, $IJ = K$. By the associative law,

$$IK = IIJ = -J, \quad KJ = IJJ = -I, \quad KI = K(-KJ) = J,$$

$$JI = KII = -K, \quad JK = J(-JI) = I.$$

For any numbers a, b, c, d,

$$Q = a + bI + cJ + dK, \qquad Q' = a - bI - cJ - dK$$

are called *conjugate quaternions*. Prove that their product in either order is $N = a^2 + b^2 + c^2 + d^2$. When a, b, c, d are real numbers, Q is called a *real* quaternion; then $N = 0$ only when a, b, c, d are all zero, so that $Q = 0$. Hence every real quaternion $Q \neq 0$ has the inverse $(1/N)Q'$, which is denoted by Q^{-1}. As at the end of Problem 4, both kinds of division by $Q \neq 0$ are possible and unique.

6. For the matrix M in (40), verify that

$$M^2 - (a+d)M + (ad - bc)S_1 = 0.$$

7. If A is the matrix (26) and if

$$X = \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} \qquad \begin{Bmatrix} k_1 \end{Bmatrix}$$

are matrices having a single column, show that equations (25) are equivalent to $AX = P$. If $m = n$ and if the determinant of A is not zero, then $X = A^{-1}P$ gives the solution of equations (25) by matrices.

CHAPTER X

Symmetric Functions

91. Sigma Functions. A polynomial function of independent variables is called *symmetric* in them if it is unaltered by the interchange of any two of the variables. Similarly for the quotient of two such polynomials, which is called a rational function. For example,

$$r^2+s^2+t^2+4r+4s+4t$$

is symmetric in r, s, t. The sum of its first three terms is denoted by Σr^2, and the sum of the last three terms is designated $4\Sigma r$. Also

$$\Sigma rs=rs+rt+st, \qquad \Sigma r^2s=r^2s+r^2t+s^2r+s^2t+t^2r+t^2s,$$

$$\sum\frac{1}{r}=\frac{1}{r}+\frac{1}{s}+\frac{1}{t}, \qquad \sum\frac{r}{s}=\frac{r}{s}+\frac{s}{r}+\frac{r}{t}+\frac{t}{r}+\frac{s}{t}+\frac{t}{s},$$

while $\Sigma rst=rst$. We have now defined seven *sigma functions*. Duplicate terms are always suppressed.

The same definitions are used also when r, s, t are any distinct numbers, instead of independent variables. In case $t=s$, we understand Σr^2 to mean the value r^2+2s^2 for $t=s$ of the function Σr^2 defined above.

In § 16 we defined the n elementary symmetric functions of n variables. For $n=3$, they are Σr, Σrs and rst.

If r, s, t are the roots of

(1) $$x^3+c_1x^2+c_2x+c_3=0,$$

we know that

(2) $$\Sigma r=-c_1, \qquad \Sigma rs=c_2, \qquad rst=-c_3.$$

92. Sums of Like Powers of the Roots. The general theory to be developed is well illustrated by the case of a quadratic equation

(3) $$x^2-px+q=0.$$

Let r and s be its roots and write s_k for r^k+s^k. We already know that $s_1=r+s$ has the value p. By the definition of a root, we have

$$(4) \qquad r^2-pr+q=0, \qquad s^2-ps+q=0.$$

Addition gives $s_2-ps_1+2q=0$. Hence $s_2=p^2-2q$. To find s_3, we multiply the first relation (4) by r and the second by s. Thus

$$r^3-pr^2+qr=0, \qquad s^3-ps^2+qs=0.$$

Addition gives $s_3-ps_2+qs_1=0$, whence $s_3=p^3-3pq$.

The reader should verify similarly that

$$s_4-ps_3+qs_2=0, \qquad s_4=p^4-4p^2q+2q^2,$$
$$=0, \qquad s_5=p^5-5p^3q+5pq^2.$$

In general, consider an equation

$$(5) \qquad f(x)=x^n+c_1x^{n-1}+c_2x^{n-2}+\cdots+c_n=0$$

having the roots $r_1, \cdots, r_n$. Employ the notation

$$(6) \qquad s_k=\Sigma r_1{}^k=r_1{}^k+r_2{}^k+\cdots+r_n{}^k.$$

From (5) we find by multiplication by x^{k-n}, where $k\geqq n$, that

$$(7) \qquad x^k+c_1x^{k-1}+c_2x^{k-2}+\cdots+c_nx^{k-n}=0.$$

This holds if we take $x=r_1, \cdots, x=r_n$ in turn. Addition gives

$$(8) \qquad s_k+c_1s_{k-1}+c_2s_{k-2}+\cdots+c_ns_{k-n}=0 \qquad (k\geqq n).$$

If $k=n$, equation (7) is the given equation (5), so that the final term in (7) is simply c_n. Hence the final term of (8) should be $c_n\cdot n$. In other words, s_0 should be n, and this is true by (6) with $k=0$ since $r_1^0=1$, etc.

Let $n=3$. Taking $k=3, k=4, \cdots$ in turn in (8), we get

$$(9) \qquad s_3+c_1s_2+c_2s_1+3c_3=0, \quad s_4+c_1s_3+c_2s_2+c_3s_1=0, \cdots$$

We know that $s_1=-c_1$ by (2). We need the value of s_2 before we can employ equations (9) in turn to compute $s_3, s_4, \cdots$. But

$$c_1^2=(r+s+t)^2=r^2+s^2+t^2+2(rs+rt+st)=s_2+2c_2,$$

by (2). Hence

$$s_2=c_1^2-2c_2, \qquad s_3=-c_1^3+3c_1c_2-3c_3.$$

For $n > 3$ we cannot employ formula (8) to compute $s_n, s_{n+1}, \cdots$ unless we know the values of $s_2, \cdots, s_{n-1}$. To find the latter we need a new formula which we proceed to derive. We start with the factored form of (5), viz.,

$$f(x) \equiv (x-r_1)(x-r_2)\cdots(x-r_n). \tag{10}$$

By the rule IV of § 45 for the derivative of a product, we obtain $f'(x)$ by multiplying the derivative $(=1)$ of a factor in (10) by the product of the remaining factors and adding all such results. If the chosen factor is $x-r_2$, the product of the remaining factors is evidently the quotient of the complete product (10) by $x-r_2$. In this way we get

$$f'(x) \equiv \frac{f(x)}{x-r_1} + \frac{f(x)}{x-r_2} + \cdots + \frac{f(x)}{x-r_n}. \tag{11}$$

We compute these fractions as follows. If r is any root of (5), then $f(r) = 0$ and

$$\begin{aligned} \frac{f(x)}{x-r} &\equiv \frac{f(x)-f(r)}{x-r} \equiv \frac{x^n - r^n}{x-r} + c_1\frac{x^{n-1}-r^{n-1}}{x-r} + \cdots + c_{n-1}\frac{x-r}{x-r} \\ &\equiv x^{n-1} + rx^{n-2} + r^2x^{n-3} + \cdots + r^kx^{n-k-1} + \cdots \\ &\quad + c_1(x^{n-2} + rx^{n-3} + \cdots + r^{k-1}x^{n-k-1} + \cdots) \\ &\quad + c_2(x^{n-3} + \cdots + r^{k-2}x^{n-k-1} + \cdots) \\ &\quad \cdots\cdots\cdots\cdots \\ &\quad + c_k(x^{n-k-1} + \cdots) + \cdots \end{aligned}$$

by actual division or by Problem 19 of § 9. Hence

$$\begin{aligned} \frac{f(x)}{x-r} &\equiv x^{n-1} + (r+c_1)x^{n-2} + (r^2+c_1r+c_2)x^{n-3} + \cdots \\ &\quad + (r^k + c_1r^{k-1} + c_2r^{k-2} + \cdots + c_{k-1}r + c_k)x^{n-k-1} + \cdots \end{aligned} \tag{12}$$

Taking r to be $r_1, \cdots, r_n$ in turn in (12), adding the results, and applying (11), we obtain

$$\begin{aligned} f'(x) &\equiv nx^{n-1} + (s_1+nc_1)x^{n-2} + (s_2+c_1s_1+nc_2)x^{n-3} + \cdots \\ &\quad + (s_k + c_1s_{k-1} + c_2s_{k-2} + \cdots + c_{k-1}s_1 + nc_k)x^{n-k-1} + \cdots \end{aligned}$$

But the derivative $f'(x)$ of (5) may be found by the rules in § 45, which give

$$f'(x)=nx^{n-1}+(n-1)c_1x^{n-2}+(n-2)c_2x^{n-3}+\cdots+(n-k)c_kx^{n-k-1}+\cdots$$

Since our two expressions for $f'(x)$ are term by term identical, we have

$$(13)\qquad \begin{cases} s_1+c_1=0, \qquad s_2+c_1s_1+2c_2=0, \cdots, \\ s_k+c_1s_{k-1}+c_2s_{k-2}+\cdots+c_{k-1}s_1+kc_k=0 \qquad (k\leqq n-1). \end{cases}$$

The two relations in the first line of (13) are of course the cases $k=1$ and $k=2$ of the general relation in the second line of (13).

Relations (8) and (13) are together called *Newton's identities*. They enable us to compute $s_1, s_2, s_3, \cdots$ in turn as functions of $c_1, c_2, \cdots$.

PROBLEMS

For equation (5) with $n\geqq 4$, compute

1. s_2, *Ans.* $s_2=c_1^2-2c_2$. **2.** s_3, *Ans.* $s_3=-c_1^3+3c_1c_2-3c_3$.

3. s_4, *Ans.* $s_4=c_1^4-4c_1^2c_2+4c_1c_3+2c_2^2-4c_4$.

4. For (5) with $n=3$, $s_4=c_1^4-4c_1^2c_2+4c_1c_3+2c_2^2$.

5. Why may we deduce Problem 4 from Problem 3 by taking $c_4=0$?

6. When $c_n\neq 0$, we may extend the definition (6) to negative values of k. In equation (5) replace x by $1/z$ and clear of fractions. If $z_1, \cdots, z_n$ are the roots of the resulting equation in z, write S_k for Σz_1^k. Find S_1, S_2, S_3 by Newton's identities. But $s_{-k}=S_k$. Hence we have found s_{-1}, s_{-2}, s_{-3}.

7. For $6x^2-5x+1=0$, $s_{-1}=5$, $s_{-2}=13$, $s_{-3}=35$.

8. For $6x^3-11x^2+6x-1=0$, $s_{-1}=6$, $s_{-2}=14$, $s_{-3}=36$.

9. For $x^3=1$, $s_1=s_{-1}=0$, $s_2=s_{-2}=0$, $s_3=s_{-3}=3$.

10. The proof which led to (8) holds also if $k<n$ and gives

$$s_k+\cdots+c_{k-1}s_1+nc_k+c_{k+1}s_{-1}+\cdots+c_ns_{-n+k}=0.$$

Subtracting (13), we get

$$(14)\qquad (n-k)c_k+c_{k+1}s_{-1}+\cdots+c_ns_{-n+k}=0 \qquad (k<n).$$

11. Taking $k=n-1$, $k=n-2$, $k=n-3$ in (14), we get

$$c_{n-1}+c_ns_{-1}=0,\ 2c_{n-2}+c_{n-1}s_{-1}+c_ns_{-2}=0,\ 3c_{n-3}+c_{n-2}s_{-1}+c_{n-1}s_{-2}+c_ns_{-3}=0.$$

Hence compute s_{-1}, s_{-2}, s_{-3} and check by Problem 6.

12. Verify that

$$s_2=\begin{vmatrix} c_1 & 1 \\ 2c_2 & c_1 \end{vmatrix},\quad s_3=-\begin{vmatrix} c_1 & 1 & 0 \\ 2c_2 & c_1 & 1 \\ 3c_3 & c_2 & c_1 \end{vmatrix},\quad s_4=\begin{vmatrix} c_1 & 1 & 0 & 0 \\ 2c_2 & c_1 & 1 & 0 \\ 3c_3 & c_2 & c_1 & 1 \\ 4c_4 & c_3 & c_2 & c_1 \end{vmatrix}.$$

93. Further Results. The preceding problems furnish illustrations of the following important general result.

THEOREM 1. *Any polynomial which is symmetric in the roots of an equation is expressible as a polynomial function of the coefficients. Likewise when "polynomial" is replaced by "rational function."*

This is proved in the Appendix.

THEOREM 2. *If* P *is a rational function of the roots* $r_1, \cdots, r_n$ *of an equation* $f(x)=0$ *of degree* n *and if* P *is symmetric in* n—1 *of the roots, say* $r_2, \cdots, r_n$, *then* P *is expressible as a rational function of* r_1 *and the coefficients of* f(x) *and of* P.

Proof. Since P is symmetric in all the roots of

$$\frac{f(x)}{x-r_1}=0,$$

Theorem 1 shows that P is expressible in terms of the coefficients of this equation and those of P.

For use in the problems, note that (11) for $x=m$ gives

$$\sum \frac{1}{r_1-m}=\frac{-f'(m)}{f(m)}. \tag{15}$$

EXAMPLE. If r, s, t are the roots of

$$f(x)=x^3-ax^2+bx-c=0, \tag{16}$$

compute

$$\sum \frac{s^2+t^2}{s+t}=\frac{s^2+t^2}{s+t}+\frac{r^2+t^2}{r+t}+\frac{r^2+s^2}{r+s}.$$

Solution. By (2), we have

$$\Sigma r=a, \qquad \Sigma rs=b, \qquad rst=c. \tag{17}$$

By the preceding Problem 1, $s^2+t^2=a^2-2b-r^2$. Also, $s+t=a-r$. Hence

$$\frac{s^2+t^2}{s+t}=\frac{a^2-2b-r^2}{a-r}=r+a+\frac{2b}{r-a}.$$

By relation (15) for $m=a$, we get

$$\sum \frac{1}{r-a}=\frac{-a^2-b}{ab-c}.$$

Hence

$$\sum \frac{s^2+t^2}{s+t}=a+3a+\frac{2b(a^2+b)}{c-ab}=\frac{2b^2-2a^2b+4ac}{c-ab}.$$

PROBLEMS

[In Problems 1–12, r, s, t are the roots of (16).]

Using $st+r(s+t)=b$ in preference to $st=c/r$, find

1. $\sum \frac{st+r^2}{s+t}$, *Ans.* $\frac{a^4-3a^2b+5ac+b^2}{ab-c}$.

2. $\sum \frac{3st-2r^2}{s+t-r}$, *Ans.* $\frac{(5a^2-12b)(a^2-4b)}{4(4ab-8c-a^3)}+\frac{13a}{4}$.

3. Using $st=c/r$, find $\sum \frac{s^2+t^2}{st+d}$.

4. Find $\Sigma(s+t)^2$.

5. Find $\sum \frac{(s-t)^2}{(s+t)^2}$.

6. To find the cubic equation having the roots $st-1/r$, $rt-1/s$ and $rs-1/t$ why do you make in (16) the substitution

$$\frac{c}{x}-\frac{1}{x}=y?$$

Find the substitution which replaces (16) by an equation with the roots

7. $2rs$, $2rt$, $2st$.

8. $\frac{3}{r}, \frac{3}{s}, \frac{3}{t}$.

9. $rs+rt$, $rt+st$, $rs+st$.

10. $r^2+rs+s^2, \cdots, s^2+st+t^2$, *Ans.* $a^2-b-ax=y$.

11. r^2+s^2, etc.

12. $\frac{r-3}{s+t-r}$, etc.

If r, s, t, u are the roots of $x^4-ax^3+bx^2-cx+d=0$, find by (15)

13. $\sum \frac{s^2+t^2+u^2}{s+t+u}$, *Ans.* $\frac{2b(c-2ab-a^3)}{a^2b-ac+d}+5a$.

14. $\sum \frac{st+su+tu}{s+t+u}$.

15. $\sum \frac{st+su+tu}{s+t+u-4}$.

16. Find $\sum \frac{s}{r}=\sum \frac{s+t+u}{r}$, *Ans.* $\frac{ac}{d}-4$.

CHAPTER XI

Elimination, Resultants, and Discriminants

94. Definitions and Examples. If the two equations

$$(1) \qquad ax+b=0, \qquad kx+l=0 \qquad (a\neq 0, \quad k\neq 0)$$

are satisfied by the same value of x, then

$$x=-\frac{b}{a}=-\frac{l}{k},$$

so that $bk=al$, and conversely. Thus the equations have a common root if and only if $bk=al$. This condition is said to be obtained from equations (1) by *eliminating* x. It may be written as $E=0$, where E denotes any of the functions $bk-al$, $al-bk$, $3bk-3al$, etc.

In general, we seek a polynomial E in the coefficients of any two equations such that $E=0$ is a necessary and sufficient condition that the equations shall have a common root. We shall call E an *eliminant* of the two equations.

Example. Find an eliminant for the equations

$$(2) \qquad f(x)=ax^2+bx+c=0, \qquad g(x)=jx^2+kx+l=0 \quad (a\neq 0).$$

Solution. Let r and s be the roots of $f=0$. By (3) of Chapter II,

$$(3) \qquad r+s=-\frac{b}{a}, \qquad rs=\frac{c}{a}.$$

There will be a common root of equations (2) if and only if the product $g(r)g(s)$ is zero. By actual multiplication this product is

$$j^2r^2s^2+jkrs(r+s)+jl(r^2+s^2)+k^2rs+kl(r+s)+l^2.$$

To avoid fractions multiply its terms by a^2 and insert the values (3), noting that

$$r^2+s^2=(r+s)^2-2rs.$$

Defining E to be $a^2g(r)g(s)$, we get

$$(4) \qquad E=j^2c^2-jkbc+jl(b^2-2ac)+k^2ac-klab+l^2a^2.$$

95. Faulty Methods of Elimination, Extraneous Factors. Natural steps in the elimination of x from two cubic equations $f(x)=0$ and $g(x)=0$ consist in finding two combinations of them which reduce to quadratic equations. On the one hand we eliminate x^3. On the other hand we eliminate the constant terms and then cancel the factor x. For example, let

$$(5) \qquad f(x)=x^3-2x^2-x+2=0, \qquad g(x)=x^3+px^2-x+q=0.$$

Employ the abbreviations $A=p+2$, $B=q-2$. Then

$$(6) \qquad g-f=Ax^2+B\equiv 0, \qquad \frac{qf-2g}{x}\equiv Bx^2-2(A+B)x-B=0$$

are the mentioned quadratic equations. By formula (4) we find that their eliminant is

$$F=B(A+B)^2(B+4A).$$

Hence we should expect that $f=0$ and $g=0$ have a root in common if and only if $F=0$. But this is false. The roots of $f=0$ are 1, -1, 2, while

$$g(1)=g(-1)=A+B, \qquad g(2)=B+4A.$$

Their product gives a correct eliminant E of equations (5). Thus

$$E=(A+B)^2(B+4A).$$

Hence $F=BE$ has the extraneous factor B. If $B=0$, so that $q=2$, a mere glance at equations (5) shows that they are inconsistent unless also $p=-2$, so that $A=0$. In case $B=0$ and $A\neq 0$, F is therefore not the true eliminant, since F is then zero and yet the equations have no common root. Hence our plausible method of elimination must be discarded.

Fortunately we have the following simple method of elimination which will be proved to lead to a correct eliminant.

96. Sylvester's Method of Elimination. In the simplest case of two linear equations (1), we multiply the terms b and l by $y=1$ and obtain two linear homogeneous equations in x and $y=1$. Hence the method of solution by determinants gives

$$\begin{vmatrix} a & b \\ k & l \end{vmatrix}=0.$$

We saw that this is also a sufficient condition that equations (1) shall have a common root. Hence $al-bk$ may be taken to be an eliminant of equations (1).

In the more typical case of a quadratic and a linear equation

$$f(x)=ax^2+bx+c=0, \qquad g(x)=kx+l=0 \qquad (k\neq 0), \tag{7}$$

we annex the equation $xg=kx^2+lx=0$ and now have three linear homogeneous equations in x^2, x, and 1. Hence the determinant of their coefficients must be zero (§ 79):

$$\begin{vmatrix} a & b & c \\ k & l & 0 \\ 0 & k & l \end{vmatrix}=0. \tag{8}$$

When this determinant is zero, we know that the equations

$$au+bv+cw=0, \qquad ku+lv=0, \qquad kv+lw=0$$

with the same coefficients as $f=0$, $xg=0$, $g=0$, respectively, have solutions not all zero (corollary in § 82). But this does not show that solutions can be found such that $u=v^2$, $w=1$. However, this is true by Problem 1 below. Hence the determinant in (8) is an eliminant of equations (7).

Sylvester's method for two quadratic equations (2) consists in using the four equations

$$xf=0, \qquad f=0, \qquad xg=0, \qquad g=0,$$

which are linear in x^3, x^2, x, 1. The determinant of their coefficients is

$$\begin{matrix} a & b & c & 0 \\ 0 & a & b & c \\ j & k & l & 0 \\ 0 & j & k & l \end{matrix} \tag{9}$$

PROBLEMS

1. Prove that an eliminant of equations (7) is

Verify that this is the value of the determinant (8).

2. Interchange the second and third rows of determinant (9), and apply Laplace's development by rows to the new determinant.

$$\textit{Ans.}\ D=(la-jc)^2-(ka-jb)(lb-kc).$$

3. Verify that the last answer is the same as (4). This proves that the determinant D in (9) is an eliminant of $f=0$ and $g=0$.

4. Verify that this D is zero when $a=j=1$, $b=2p$, $k=4p$, $c=p^2-d^2$, $l=3p^2+2pd-d^2$, where p and d are arbitrary. Numerical cases follow.

Prove that the following pairs of equations have a root in common by exhibiting Sylvester's four equations and evaluating their determinant.

5. $x^2+2x-8=0$, $\quad x^2+4x-12=0$.
6. $x^2+2x-15=0$, $\quad x^2+4x-5=0$.
7. $x^2+x-42=0$, $\quad x^2-4x-77=0$.

We shall now consider Sylvester's method for any two equations

$$(10)\qquad f(x)=a_0x^m+\cdots+a_m=0,\qquad g(x)=b_0x^n+\cdots+b_n=0$$
$$(a_0\neq 0,\quad b_0\neq 0).$$

We employ the $n+m$ equations

$$(11)\quad x^{n-1}f=0,\ x^{n-2}f=0,\ \cdots,\ xf=0,\ f=0,\ x^{m-1}g=0,\ \cdots,\ xg=0,\ g=0,$$

which are linear and homogeneous in the $n+m$ quantities

$$(12)\qquad x^{n+m-1},\ x^{n+m-2},\ \cdots,\ x,\ 1.$$

Let D denote the determinant of their coefficients in (11).

First, let equations (10) have a common root x. By § 79, the product of D by each unknown (12) in equations (11) is zero, whence $D=0$.

Second, let $D=0$. By Theorem 14 of § 81, there exist constants $r_{n-1}, \cdots, s_0$, not all zero, such that

$$r_{n-1}x^{n-1}f+\cdots+r_1xf+r_0f+s_{m-1}x^{m-1}g+\cdots+s_1xg+s_0g\equiv 0,$$

identically in x. Expressed otherwise, $Rf+Sg\equiv 0$, where

$$R=r_{n-1}x^{n-1}+\cdots+r_1x+r_0,\qquad S=s_{m-1}x^{m-1}+\cdots+s_1x+s_0.$$

Evidently R and S are not both identically zero. If S were identically zero, then Rf would be identically zero and R not, so that $f\equiv 0$, contrary to $a_0\neq 0$. This shows that S is not identically zero.

Suppose that f and g have no common factor linear in x. Allowing for possible multiple roots of $f=0$, note that the highest power of each linear factor occurring in f divides $Sg \equiv -Rf$ and hence divides S. In other words, f divides S. But this is impossible since f is of degree m and since S has a degree $\leqq m-1$, not being identically zero. This contradiction shows the falsity of our supposition that f and g have no common linear factor.

Hence equations (10) have a common root if and only if $D=0$. Thus Sylvester's method is a perfect method of elimination since it yields a true eliminant $E=D$.

The following example and problems do not require or illustrate any of our preceding or later results and hence may be omitted.

Example. Find all sets of solutions of

Solution. Add the double of the first equation to the second. We get $21y=155+18x-5x^2$. Insert the resulting expression for y into our first equation. We obtain

Here $928=2^5 \cdot 29$. We find as usual the integral roots 1, 4, 8. The sum of all four roots must be 36/5. Hence the fourth root is $-29/5$. For each x we compute y by the equation involving $21y$. The answers are $(x, y)=(1, 8)$, $(4, 7)$, $(8, -1)$, $(-29/5, -28/5)$.

PROBLEMS

1. Find a necessary and sufficient condition that

$$f(x)=x^4+px^3+qx^2+rx+s=0$$

shall have one root the negative of another root. Hint: Use also $f(-x)=0$. *Ans.* $pqr-p^2s-r^2=0$.

2. The problem to find all points of intersection of two general conics leads by elimination of y to an equation of the fourth degree for the abscissas x.

Find the points of intersection of the conics having the equations

3. $x^2+y^2=65$, $2x^2-3y^2-18x-21y+40=0$.
4. $x^2+y^2=65$, $4x^2-y^2-21x+18y=105$, *Ans.* $(-1,8)$, $(8,1)$, $(7,4)$, $(-\frac{28}{5}, -\frac{29}{5})$.
5. $x^2+y^2=25$, $x^2+3y^2+6x=75$, *Ans.* $(0, \pm 5)$, $(3, \pm 4)$.
6. $x^2+y^2=25$, $x^2+3y^2+19x=0$.
7. $x^2+y^2=65$, $xy=8$, *Ans.* $(\pm 1, \pm 8)$, $(\pm 8, \pm 1)$.
8. Find the result of eliminating s and t between the equations

$$r+s+t=a, \qquad rs+rt+st=b, \qquad rst=c.$$

9. Eliminate y between the equations $y^3=v$ and $x=y^2+ry$ to get $x^3-3rvx-v^2-r^3v=0$. Choose r and v so that this shall be identical with $x^3+px+q=0$. Hence solve the latter (Euler, 1764).

10. Eliminate y between $y^3=v$ and $x=y^2+ey+f$ and get

$$\begin{vmatrix} 1 & e & f-x \\ e & f-x & v \\ f-x & v & ev \end{vmatrix}$$

Since this cubic equation in x can be identified with the general cubic equation by choice of e, f, v, the process solves the latter.

97. Resultants. An eliminant E may be multiplied by any numerical constant without disturbing the condition $E=0$ for a common root of two equations. Also, if E has a factor F^k, we may replace that factor by F^r, where r is any integer exceeding zero. We shall now select a definite E and call it the resultant. Consider two polynomials

$$(13) \quad f(x)=a_0x^m+\cdots+a_m, \quad g(x)=b_0x^n+\cdots+b_n \quad (a_0\neq 0,\ b_0\neq 0),$$

such that $f(x)=0$ is known to have m (complex) roots $r_1, \cdots, r_m$, not necessarily distinct. This is true by Chapters II and V when $m=1, 2, 3$, or 4, and all our problems fall under one of these cases. For a general proof see the Appendix. Evidently equations (13) have a root in common if and only if

$$g(r_1)\,g(r_2)\,\cdots\,g(r_m)=0.$$

To be rid of denominators when we evaluate this product as a function of $a_0, \cdots, b_n$, it suffices to multiply the product by a_0^n (see the example in § 94 and the end of § 16). We therefore define the resultant of f and g to be

$$(14) \qquad R(f, g)=a_0^n\,g(r_1)\,g(r_2)\cdots g(r_m),$$

or preferably the equivalent polynomial in $a_0, \cdots, b_n$.

For example, we have proved that the resultant of the functions (2) is the determinant (9) whose value is the polynomial (4). For the functions f and g in (7), it was shown in Problem 1 of the first set of problems in § 96 that $R(g, f)$ is the determinant (8).

PROBLEMS

1. Prove that $R(ax+b,\ kx+l)=al-bk$.
2. $R(ax^2+bx+c,\ kx+l)=al^2-blk+ck^2$.
3. $R(f,\ gh)=R(f,\ g)\ R(f,\ h)$.
4. Using the factored forms of f and g in (13), prove that

$$R(f,\ g)=(-1)^{mn}\ R(g,f).$$

5. $R(a_0x^m,\ g)=a_0^n b_n^m$. Hence the latter is a term of $R(f,\ g)$.
6. Show that $R(f,\ g)$ is homogeneous and of total degree m in $b_0,\ b_1,\ \cdots,$
7. Hence prove that $R(f,\ g)$ is homogeneous and of total degree n in $a_0,\ \quad,\ a_m$.

98. Sylvester's Determinants Are Resultants. We have already proved that any Sylvester determinant is an eliminant. The proof that it is actually a resultant is the same as that given for the following typical case. Consider the equations

$$(15) \qquad f(x)=ax^3+bx^2+cx+d=0, \quad g(x)=jx^2+kx+l=0$$

The determinant of the coefficients of x^4, x^3, x^2, x, 1 in Sylvester's equations

$$xf=0, \qquad f=0, \qquad x^2g=0, \qquad xg=0 \qquad g=0$$

is evidently

$$(16) \qquad D=\begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ j & k & l & 0 & 0 \\ 0 & j & k & l & 0 \\ 0 & 0 & j & k & l \end{vmatrix}.$$

To prove that D is the resultant $R(g,f)$, consider the equation

$$(17) \qquad \begin{vmatrix} a & b & c & d-z & 0 \\ 0 & a & b & c & d-z \\ j & k & l & 0 & 0 \\ 0 & j & k & l & 0 \\ 0 & 0 & j & k & l \end{vmatrix}=0.$$

Laplace's development of determinant (17) by rows gives

in which the value of p is not needed in the proof, while the constant term is the value (16) of determinant (17) for $z=0$. Denote the roots of $g(x)=0$ by r_1 and r_2, and let r denote either root. We shall evaluate determinant (17) when $z=f(r)$. To the elements of the last column we add the products of the elements of the first four columns by r^4, r^3, r^2, r, respectively. Then the elements of the new last column, read from the bottom upwards, are respectively

$$g(r),\ rg(r),\ r^2g(r),\ \Delta=ar^3+br^2+cr+d-f(r)=f(r)-f(r),\ r\Delta,$$

which are all zero. Hence determinant (17) vanishes for $z=f(r_1)$ and for $z=f(r_2)$. In other words, $f(r_1)$ and $f(r_2)$ are the roots of equation (18). Since the product of its roots is equal to D/j^3, we see that

$$D=j^3\, f(r_1)\, f(r_2).$$

Hence by the definition (14), $R(g,f)=D$. By the preceding Problem 4, $R(f,g)=R(g,f)$. Thus determinant (16) is the resultant $R(f,g)$.

PROBLEMS

1. Evaluate determinant (16) as follows. Multiply its elements in the third and fourth rows by a to get a^2D. In the new determinant add to the elements of the third row the products of the elements of the first row by $-j$, those of the second row by $-k$, and those of the fourth row by b/a. To the elements of the fourth row add the products of those of the second row by $-j$. We get

$$a^2D=\left|\begin{array}{cc|ccc|}
a & b & c & d & 0\\
0 & a & b & c & d\\
0 & 0 & al-cj & bl-ck-dj & -dk\\
0 & 0 & ak-bj & al-cj & -dj\\
0 & 0 & j & k & l
\end{array}\right|$$

By Laplace's development by rows we see that D is equal to the 3-rowed determinant enclosed by dotted lines.

Prove that the following equations have a root in common by verifying that the 3-rowed determinant in Problem 1 is zero.

2. x^3

3.

4. Evaluate the resultant $R(f, g)$ of the polynomials (13) when $m=n=3$ by the method used for Problem 1. By Sylvester's method for the products of f and g by x^2, x, and 1, we get

$$\begin{array}{cccccc}
 & & & & 0 & 0 \\
0 & a_0 & a_1 & a_2 & a_3 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & a \\
b_0 & b_1 & b_2 & b_3 & 0 & 0 \\
0 & b_0 & b_1 & b_2 & b_3 & 0 \\
0 & 0 & b_0 & b_1 & b_2 & b
\end{array}$$

To the products of the elements of the fourth row by a_0 add the products of the elements of the first, second, third, fifth, sixth rows by $-b_0$, $-b_1$, $-b_2$, a_1, a_2, respectively. To the product of the elements of the fifth row by a_0 add the products of the elements of the second, third, sixth rows by $-b_0$, $-b_1$, a_1, respectively. Finally, to the products of the elements of the sixth row by ( ) add the products of the elements of the third row by $-b_0$. We get

$$\begin{array}{ccc|ccc|}
a_0 & a_1 & a_2 & a_3 & 0 & 0 \\
0 & a_0 & a_1 & a_2 & a_3 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & a_3 \\
0 & 0 & 0 & (a_0b_3) & (a_1b_3) & (a_2b_3) \\
0 & 0 & 0 & (a_0b_2) & (a_0b_3)+(a_1b_2) & (a_1b_3) \\
0 & 0 & 0 & (a_0b_1) & (a_0b_2) & (a_0b_3)
\end{array}$$

in which (a_ib_j) denotes $a_ib_j-a_jb_i$. By Laplace's development by rows we see that R is equal to the 3-rowed determinant enclosed by dotted lines.

By evaluating the preceding 3-rowed determinant, prove that the following equations have a root in common.

5.

6. x^3- [illegible]

7. Solve $f(x)=x^3+x^2-41x-105=0$, given that two roots r and s are such that $r-2s=1$. Hint: $f(x)$ and $f(1+2x)$ have the common factor $x-s$.

99. Imaginary Roots. Equations of degree $\leqq 4$ may be treated as follows.

EXAMPLE 1. Find the imaginary roots of

$$z^4-z^2+4z+56=0. \tag{19}$$

Solution. Put $z=x+yi$, expand, and equate the real part to zero, and likewise the pure imaginary part. We get

$$(20) \qquad x^4-6x^2y^2+y^4 \text{ [illegible] } 0, \qquad 2y(2x^3-2xy^2-x+2)=0.$$

Since z shall be imaginary, $y\neq 0$. Thus the quantity in parentheses is zero, so that $x\neq 0$ and

$$(21) \qquad y^2=\frac{2x^3-x+2}{2x}.$$

Elimination of y^2 from the first equation (20) gives

$$(22) \qquad -16x^6+8x^4+223x^2+4=0.$$

The corresponding cubic equation in $w=x^2$ is seen to have the integral root 4. The quotient of the function (22) by x^2-4 is $-16x^4-56x^2-1$, which is negative for every real x. Hence the only real roots of (22) are ± 2. By formula (21), we get $y^2=4$ when $x=2$, and $y^2=3$ when $x=-2$. Hence the imaginary roots of (19) are $2\pm 2i$, $-2\pm\sqrt{3}i$.

In general, for any equation $f(z)=0$ with real coefficients, we expand $f(x+yi)$ by Taylor's formula and get

$$f(x)+f'(x)yi-f''(x)\frac{y^2}{1\cdot 2}-f'''(x)\frac{y^3i}{1\cdot 2\cdot 3}+\cdots=0.$$

Since x and y are to be real, and $y\neq 0$, the real part must be zero, and likewise the imaginary part. Hence

$$(23) \qquad \begin{cases} f(x)-f''(x)\dfrac{y^2}{1\cdot 2}+f''''(x)\dfrac{y^4}{1\cdot 2\cdot 3\cdot 4}-\cdots=0, \\[2ex] f'(x)-f'''(x)\dfrac{y^2}{1\cdot 2\cdot 3}+f^{(5)}(x)\dfrac{y^4}{5!}-\cdots=0. \end{cases}$$

The resultant $R(x)$ of these equations in the unknown y^2 must be zero. For each root of $R(x)=0$ we seek the corresponding root y^2 of either equation (23).

When $f(z)$ is of degree 3 or 4, the second equation (23) involves y^2, but not y^4, etc. Proceed as for equation (19).

EXAMPLE 2. For $f(z)=z^4-z+1$, equations (23) are

$$x^4-x+1-6x^2y^2+y^4=0, \qquad 4x^3-1-4xy^2=0.$$

Thus

$$y^2=x^2-\frac{1}{4x}, \qquad -4x^6+x^2+\frac{1}{16}=0.$$

The cubic equation in x^2 has the single real root

$$x^2=0.528727, \qquad x=\pm 0.72714.$$

Then $y^2=0.184912$ or 0.87254, and

$$z=x+yi=0.72714\pm 0.43001i, \qquad -0.72714\pm 0.93409i.$$

PROBLEMS

Find the imaginary roots of

1. $z^4-4z^3+9z^2-16z+20=0$. Hint:

$$E(x)\equiv x(x-2)(16x^4-64x^3+136x^2-144x+65)=0,$$

and the last factor becomes $(w^2+1)(w^2+9)$ for $2x=w+2$. *Ans.* $2\pm i$, $\pm 2i$.

2. $z^4-6z^3+19z^2-54z+90=0$.
3. $z^4+3z^2+28z+78=0$.
4. $z^4-15z^2+52z-42=0$.
5. $z^4-4z^3+11z^2-14z+10=0$, *Ans.* $1\pm i$, $1\pm 2i$.
6. $z^4-4z^2+8z-4=0$.
7. $z^4+z^2-2z+6=0$.
8. $z^4-23z^2+54z+22=0$.
9. $z^4+2z^2-32z+65=0$, *Ans.* $2\pm i$, $-2\pm 3i$.
10. $z^6+3z^4+32z^3+67z^2+32z+65=0$, *Ans.* $2\pm 3i$, $-2\pm i$, $\pm i$.
11. $z^6-2z^5-5z^4+16z^3-16z^2+4=0$.

100. Discriminants. Let $f(x)=ax^m+\cdots$ be a polynomial of degree m whose factored form is

$$f(x)\equiv a(x-r_1)(x-r_2)\cdots(x-r_m). \tag{24}$$

The *discriminant* of f is defined to be a^{2m-2} times the product of the squares of the differences of $r_1, \cdots, r_m$. As in § 35, the chosen power of a is the lowest power of a which eliminates fractions when we express D as a polynomial in the coefficients of $f(x)$.

Differentiating (24), we see as below (10) of § 92 that

$$f'(r_1)=a(r_1-r_2)(r_1-r_3)\cdots(r_1-r_m),$$
$$f'(r_2)=a(r_2-r_1)(r_2-r_3)\cdots(r_2-r_m),$$
$$\cdots\cdots\cdots\cdots$$
$$f'(r_m)=a(r_m-r_1)(r_m-r_2)\cdots(r_m-r_{m-1}).$$

In the second line replace r_2-r_1 by $-(r_1-r_2)$. In $f'(r_3)$ replace $(r_3-r_1)(r_3-r_2)$ by $(-1)^2(r_1-r_3)(r_2-r_3)$. In the last line replace each r_m-r_i by $-(r_i-r_m)$. Multiplication now gives

$$a^{m-1}f'(r_1)\cdots f'(r_m)=(-1)^{1+2+\cdots+m-1}\,aD.$$

The sum in the exponent is equal to $\frac{1}{2}m(m-1)$. By (14), the left member is the resultant of $f(x)$ and $f'(x)$. This proves

$$D=(-1)^{\frac{1}{2}m(m-1)}\frac{1}{a}R(f,f'). \tag{25}$$

Another method to find D is given in Problem 3 below; it is next illustrated by the case $m=3$.

EXAMPLE. Find the discriminant D of $f(x)=x^3+px+q$.

Solution. Let a, b, c denote the roots of $f=0$. By § 71,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \equiv (b-a)(c-a)(c-b).$$

Write s_i for $a^i+b^i+c^i$, Then

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}.$$

The second determinant is equal to the first. Hence their product is equal to D. By Problems 1–3 of § 92, we have

$$s_1=0, \qquad s_2=-2p, \qquad s_3=-3q, \qquad s_4=2p^2$$

$$D=\begin{vmatrix} 3 & 0 & -2p \\ 0 & -2p & -3q \\ -2p & -3q & 2p^2 \end{vmatrix} = -4p^3-27q^2.$$

PROBLEMS

1. By Problem 1 of § 98, show that the discriminant of ax^3+bx^2+cx+d is

$$\frac{-1}{a}\begin{vmatrix} -2ac & -bc-3ad & -2bd \\ -ab & -2ac & -3ad \\ 3a & 2b & c \end{vmatrix}$$

$$=18abcd-4b^3d+b^2c^2-4ac^3-27a^2d^2.$$

2. Prove that the discriminant of the product of two functions is equal to the product of their discriminants multiplied by the square of their resultant. Hint: Use the expressions in terms of the differences of the roots.

3. For $a=1$, show that the discriminant is equal to

$$\begin{vmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^{m-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & r_m & r_m^2 & \cdots & r_m^{m-1} \end{vmatrix}^2 = \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{m-1} \\ s_1 & s_2 & s_3 & \cdots & s_m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{m-1} & s_m & s_{m+1} & \cdots & s_{2m-2} \end{vmatrix},$$

where $s_1 = r_1^t + \cdots + r_m^t$. See the above example.

4. Hence find the discriminant of x^4+3Cx^2+dx+e.

Ans. $4(4e+3C^2)^3-27(8Ce-d^2-2C^3)^2$.

CHAPTER XII

Roots of Unity and Regular Polygons

101. Roots of Unity. In Chapter I we saw that the cube roots of unity are

$$(1) \qquad 1, \quad \omega=\cos 120°+i \sin 120°,\ \omega^2=\cos 240°+i \sin 240° \ (\omega^3=1),$$

and that the fourth roots of unity are

$$(2) \qquad i=\cos 90°+i \sin 90°, \quad i^2=-1, \quad i^3=-i, \quad i^4 \ (i^4=1).$$

Henceforth we shall usually measure angles in radians (an angle of 180 degrees being equal to π radians, where $\pi=3.1416$, approximately). Then

$$(3) \qquad R=\cos\frac{2\pi}{n}+i \sin\frac{2\pi}{n}$$

is an nth root of unity since $R^n=\cos 2\pi+i \sin 2\pi=1$ by De Moivre's theorem (§ 4). For every integer k, R^k is an nth root of unity since $(R^k)^n=(R^n)^k=1$. Thus

$$(4) \qquad R,\ R^2,\ R^3,\ \cdots,\ R^{n-1},\ R^n \qquad (R^n=1)$$

are all nth roots of unity. If two of them were equal, we would find by cancellation that $R^s=1$, where $1 \leqq s \leqq n-1$. By (3) and De Moivre's theorem,

$$(5) \qquad 1=R^s=\frac{2\pi s}{n}\ \ldots\ 2\pi s \qquad 2\pi s \qquad \frac{2\pi s}{n}$$

The last relation shows that $2\pi s/n$ is a multiple $m\pi$ of π. Then shall $\cos m\pi=1$, whence m is an even integer, say $2t$. Hence $2s/n=m=2t$, $s=nt$. This contradicts $1 \leqq s \leqq n-1$. Our assumption that two of the numbers (4) are equal is therefore false. By § 13, $x^n=1$ has at most n roots. We have therefore proved

THEOREM 1. *The* n *numbers* (4) *are distinct and give all the* n*th roots of unity.*

One nth root of any complex number $r(\cos A+i \sin A)$ is the product of the real nth root ρ of the positive real number r by $t=\cos A/n+i \sin A/n$. All the nth roots are the products of one such root ρt by the numbers R^k in (4), since $(\rho tR^k)^n=rt^n=r(\cos A+i \sin A)$. We use the value (3) of R and form the product tR^k as in § 3. This proves

THEOREM 2. *The* n*th roots of* r(cos A+i sin A) *are*

$$\rho\left[\cos\left(\frac{A+2k\pi}{n}\right)+i \sin\left(\frac{A+2k\pi}{n}\right)\right] \qquad (k=0, 1, \cdots, n-1), \tag{6}$$

where ρ *is the real* n*th root of* r.

PROBLEMS

1. Show that the numbers (1), (2), and (4) are represented (§ 3) by the vertices of an equilateral triangle, square, and regular polygon of n sides, respectively. In particular, this gives another proof of Theorem 1.

2. When $n=6$, $R=-\omega^2$. The sixth roots of unity are the three cube roots of unity and their negatives. Check by factoring x^6-1.

3. Which powers of a ninth root (3) of unity are cube roots of unity?

4. Find the fifth roots of -32.

102. Primitive Roots of Unity. By (2), i^4 is the lowest power of i which gives unity, so that i will be called a primitive fourth root of unity. Another one is $-i$. While i^2 (or -1) is a fourth root of unity, it is not primitive since $(i^2)^2=1$.

We make the following general definition. An nth root of unity is called *primitive* if n is the smallest positive integral exponent of a power of it that is equal to unity. Expressed otherwise, r is a primitive nth root of unity if and only if $r^n=1$ and $r^t \neq 1$ for all positive integers t less than n.

We proved that the numbers (4) are distinct, so that only the last one is unity. Hence R, defined by (3), is a primitive nth root of unity. We proved that these numbers (4) give all the nth roots of unity. Hence if we desire all the primitive nth roots of unity, we must look for them among the numbers (4); the question as to which ones are to be chosen is answered by the next theorem. We recall that two integers are called relatively prime if they have no common divisor >1.

THEOREM 3. *The primitive* n*th roots of unity are precisely those of the numbers* (4) *whose exponents are relatively prime to* n.

Proof. If k and n have a common divisor $d(d>1)$, R^k is not a primitive nth root of unity, since

$$(R^k)^{\frac{n}{d}} = (R^n)^{\frac{k}{d}} = 1,$$

and the exponent n/d is a positive integer less than n.

But if k and n are relatively prime, R^k is a primitive nth root of unity. To prove this, we must show that $(R^k)^t \neq 1$ if t is a positive integer $<n$. By De Moivre's theorem,

$$R^{kt} = \cos\frac{2kt\pi}{n} + i\sin\frac{2kt\pi}{n}.$$

If this were equal to unity, kt would be a multiple of n, as proved below (5) with $s=kt$. Since the first factor k is prime to n, the second factor t would be a multiple of n. This contradicts our assumption that $0<t<n$.

EXAMPLE. Show that the six primitive fourteenth roots of unity are the negatives of the primitive seventh roots of unity.

Solution. For $n=14$, Theorem 3 shows that the primitive fourteenth roots of unity are R^j, $1 \leqq j \leqq 13$, where j is odd and $j \neq 7$. But $R^2, R^4, R^6, R^8, R^{10}, R^{12}$ are evidently seventh roots of unity and are primitive since no exponent is divisible by 7. For $x=R^7$, we have $x^2-1=0$, $x-1\neq 0$, whence $x+1=0$, so that $R^7=-1$. Thus

$$R^{13}=-R^6,\quad R^{11}=-R^4,\quad R^9=-R^2,\quad R^5=-R^{12},\quad R^3=-R^{10},\quad R=-R^8,$$

which prove the statement in the example.

PROBLEMS

1. Show that the primitive cube roots of unity are ω and ω^2.

2. For R given by (3), prove that the primitive nth roots of unity are (i) for $n=6$, R, R^5; (ii) for $n=8$, R, R^3, R^5, R^7; (iii) for $n=12$, R, R^5, R^7, R^{11}.

3. When n is a prime, prove that any nth root of unity, other than 1, is primitive.

4. Show that the six primitive eighteenth roots of unity are the negatives of the primitive ninth roots of unity.

5. If R is a primitive fifteenth root (3) of unity, verify that R^3, R^6, R^9, R^{12} are the primitive fifth roots of unity, and R^5 and R^{10} are the primitive cube roots of unity Show that their eight products by pairs give all the primitive fifteenth roots of unity.

6. Count the primitive nth roots of unity when $n=21$.

7. Let R be a primitive nth root (3) of unity, where n is a product of two different primes p and q. Show that $R, \cdots, R^n$ are primitive with the exception of $R^p, R^{2p}, \cdots,$

R^{qp}, whose q-th powers are unity, and R^q, R^{2q}, $\cdots$, R^{pq}, whose p-th powers are unity. These two sets of exceptions have only R^{pq} in common. Hence there are exactly $pq-p-q+1$ primitive nth roots of unity.

8. Find the number of primitive nth roots of unity if n is a square of a prime p. Hint: If $n=9$, see § 104.

9. If r is any primitive nth root of unity, prove that $r, r^2, \cdots, r^n$ are distinct and give all the nth roots of unity. Of these show that r^k is a primitive nth root of unity if and only if k is relatively prime to n.

103. Regular Polygon of Seven Sides and Seventh Roots of Unity. If

$$R=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}, \tag{7}$$

we have seen that $R, R^2, \cdots, R^6, R^7$ ($R^7=1$) give all the roots of $y^7=1$ and are complex numbers represented by the vertices of a regular polygon of seven sides inscribed in a circle whose radius is unity and whose center is the origin of coordinates.

The product of the value of R by $\cos(2\pi/7)-i\sin(2\pi/7)$ is the sum of the squares of the cosine and sine of $2\pi/7$ and hence is unity. Thus

$$\frac{1}{R}=\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}, \quad R+\frac{1}{R}=2\cos\frac{2\pi}{7}. \tag{8}$$

By using rather artificial devices from trigonometry, we found (§ 29) a cubic equation having $2\cos(2\pi/7)$ as one root. We shall derive this equation by a method which will illustrate some useful general principles.

From y^7-1 we remove the factor $y-1$ and conclude that

(9)

has the roots $R, R^2, \cdots, R^6$. The desired cubic equation has the root $R+1/R$ by (8). Hence it is a natural step to make the substitution

$$y+\frac{1}{y}=x \tag{10}$$

in (9). After dividing its terms by y^3, we have

$$\left(y^3+\frac{1}{y^3}\right)+\left(y^2+\frac{1}{y^2}\right)+\left(y+\frac{1}{y}\right)+1=0. \tag{11}$$

By squaring and cubing the members of (10), we see that

$$y^2+\frac{1}{y^2}=x^2-2, \qquad y^3+\frac{1}{y^3}=x^3-3x. \tag{12}$$

Substituting these values into (11), we obtain

$$x^3+x^2-2x-1=0. \tag{13}$$

That is, the substitution (10) converts equation (9) into (13).

If in (10) we assign to y the six values $R, \cdots, R^6$, we obtain only three distinct values of x:

$$x_1=R+\frac{1}{R}=R+R^6, \quad x_2=R^2+\frac{1}{R^2}=R^2+R^5, \quad x_3=R^3+\frac{1}{R^3}=R^3+R^4. \tag{14}$$

These three numbers are therefore the roots of equation (13).

104. Regular Polygon of Nine Sides and Ninth Roots of Unity. Let

$$R=\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\cdot \tag{15}$$

Then R, R^2, R^4 R^5, R^7, R^8 give all the primitive ninth roots of unity, while R^3, R^6, and $R^9=1$ are the roots of $y^3=1$. Hence the six primitive roots are the roots of

$$\frac{y^9-1}{y^3-1}=y^6+y^3+1=0. \tag{16}$$

Divide its terms by y^3 and employ the second formula (12). We see that the substitution (10) converts equation (16) into

$$x^3-3x+1=0. \tag{17}$$

By (8) with denominators 7 replaced by 9, we see that this equation (17) has the root $2\cos(2\pi/9)=2\cos 40°$. We also obtain (17) if we take $A=120°$ in equation (1) of Chapter IV, where we proved that it is impossible to trisect angle 120° with ruler and compasses and hence is impossible to so construct a regular polygon of nine sides (an angle at the center being 40°).

105. Reciprocal Equations. An equation having the property that the reciprocal of each root is also a root is called a *reciprocal equation.* A mere inspection of (11) shows that it is a reciprocal equation, and likewise for the equivalent equation (9). Also (16) is a reciprocal equation. The same is true of $y^3+by^2-by-1=0$.

Except for small values of n, the treatment of the reciprocal equation $x^n=1$ by the above method for reciprocal equations (that is, by making the substitution (10)) is a complete waste of time, since the solution of the resulting equation in y of high degree is far more difficult than the solution of $x^n=1$ by our later methods. Having also in mind that a proposed equation is very rarely a reciprocal equation, we shall merely quote the main theorem concerning reciprocal equations (*First Course*, page 38).

THEOREM 4. *After we have removed a possible factor* y+1 *or* y−1 *or both factors from a reciprocal equation, we always find that the resulting depressed equation is equivalent to*

$$\left(y^t+\frac{1}{y^t}\right)+c_1\left(y^{t-1}+\frac{1}{y^{t-1}}\right)+\cdots+c_{t-1}\left(y+\frac{1}{y}\right)+c_t=0. \tag{18}$$

This is converted into an equation in x of degree t by the substitution (10) and formulas of type (12).

EXAMPLE. Solve $y^5-3y^4+y^3+y^2-3y+1=0$.

Solution. Removing the factor $y+1$, we get

$$y^4-4y^3+5y^2-4y+1=0,$$

$$y^2+\frac{1}{y^2}-4\left(y+\frac{1}{y}\right)+5=0, \qquad x^2-2-4x+5=0,$$

by (12) and (10). Its roots are $x=1$ and 3. For these x's, (10) becomes $y^2-y+1=0$ and $y^2-3y+1=0$. Solving these quadratic equations, we see that the roots of the proposed equation are -1, $\frac{1}{2}(1\pm i\sqrt{3})$, $\frac{1}{2}(3\pm\sqrt{5})$.

PROBLEMS

1. Compute the elementary symmetric functions of the numbers (14), recalling that R is a root of (9), and then verify (§ 16) that the numbers (14) are the roots of (13).

2. Using only § 104, show at once that the roots of (17) are

$$R+R^8, \qquad R^2+R^7, \qquad R^4+R^5. \tag{19}$$

3. Verify Problem 2 by the method for Problem 1.

4. Solve $y^5-7y^4+y^3-y^2+7y-1=0$. *Ans.* 1, $\frac{1}{2}(7\pm\sqrt{45})$, $\frac{1}{2}(-1\pm\sqrt{3}i)$.
5. Solve $y^5-4y^4+y^3+y^2-4y+1=0$. Hint: Remove the factor $y+1$.
6. Solve $y^4+4y^3-3y^2+4y+1=0$. *Ans.* $\frac{1}{2}(1\pm\sqrt{3}i)$, $\frac{1}{2}(-5\pm\sqrt{21})$.
7. Solve $y^5-1=31(y-1)^5$.
8. Solve $y^5-ay^4+by^3-by^2+ay-1=0$ by radicals.

9. There is an elegant geometrical construction which leads simultaneously to sides of an inscribed regular pentagon and decagon. The imaginary fifth roots of unity satisfy $y^4+y^3+y^2+y+1=0$, which by the substitution (10) becomes $x^2+x-1=0$. One root of the latter is $R+\frac{1}{R}=2\cos\frac{2\pi}{5}$. In a circle of radius unity and center O draw two perpendicular diameters AOA', BOB'. With the middle point M of OA' as center and radius MB draw a circle cutting OA at C (Fig. 27). Show that OC and BC are the sides s_{10} and s_5 of the inscribed regular decagon and pentagon respectively. Hints:

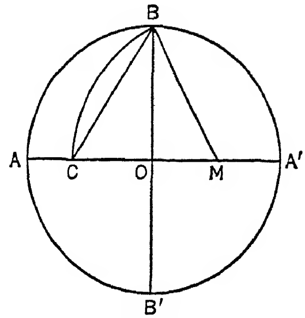


FIG. 27

$$MB=\tfrac{1}{2}\sqrt{5},\qquad OC=\tfrac{1}{2}(\sqrt{5}-1),\qquad BC=\sqrt{1+OC^2}=\tfrac{1}{2}\sqrt{10-2\sqrt{5}},$$

$$s_{10}=2\sin 18^\circ=2\cos\frac{2\pi}{5}=OC,$$

$$s_5^2=(2\sin 36^\circ)^2=2\left(1-\cos\frac{2\pi}{5}\right)=\frac{1}{4}(10-2\sqrt{5}),\qquad s_5=BC.$$

10. If s_n is the sum of the n-th powers of the roots of a reciprocal equation, then

106. Periods of Roots of Unity. For $n=7$, the three sums

(20) $$R+R^6,\qquad R^2+R^5,$$

each of two of the six imaginary seventh roots of unity, are called their *three periods each of two terms*. We meet these periods in formula (14). Similarly, there are *two periods each of three terms*:

(21) $$R+R^2+R^4,\qquad R^3+R^6+R^5.$$

Gauss discovered a general method to obtain such periods for nth roots of unity. We shall first discuss the case $n=7$. We seek a positive integer g such that $R, \cdots, R^6$ can be arranged in the order

(22) $$R,\ R^g,\ R^{g^2},\ R^{g^3},\ R^{g^4},\ R^{g^5},$$

where each term is the gth power of its predecessor. Evidently $g\neq 1$. If $g=2$, the fourth term becomes $R^8=R$, so that $g\neq 2$. If $g=3$, we get

$$R,\ R^3,\ R^2,\ R^6,\ R^4,\ R^5, \tag{23}$$

where each term is the cube of its predecessor. We have therefore reached our goal by taking $g=3$.

The periods (21) are the sums of alternative terms of (23), viz., the first, third, and fifth terms; also the second, fourth, and sixth terms.

The periods (20) are the sums of R and the third term R^6 after it, R^3 and the third term R^4 after it, R^2 and the third term R^5 after it.

PROBLEMS

1. Arrange the six primitive ninth roots of unity (§ 104) so that each term is the square of its predecessor. Then show that $R+R^4+R^7$ and $R^2+R^8+R^5$ are the two periods each of three terms. Find the three periods each of two terms.

2. When $n=13$ verify that we may take $g=2$ and deduce the three periods each of four terms. *First Ans.* $R+R^8+R^{12}+R^5$.

3. The periods (21) are the roots z_1 and z_2 of $z^2+z+2=0$. Then R, R^2, R^4 are the roots of $w^3-z_1w^2+z_2w-1=0$.

107. Regular Polygon of Seventeen Sides. Just before his nineteenth birthday, in 1796, Gauss made the remarkable discovery that it is possible to construct with ruler and compasses a regular polygon of seventeen sides. This fact had not even been suspected during the twenty centuries from Euclid to Gauss. We employ

$$R=\cos A+i\sin A, \qquad A$$

Since $R^{17}-1=0$ and $R-1\neq 0$, we have $R^{16}+\cdots+R+1=0$. As in § 106 we may take $g=3$ and arrange $R, \cdots, R^{16}$ in the order

$$R,\ R^3,\ R^9\ R^{10}\ R^{13},\ R^5,\ R^{15},\ R^{11},\ R^{16},\ R^{14},\ R^8,\ R^7,\ R^4,\ R^{12},\ R^2,\ R^6,$$

where each term is the cube of its predecessor.

Taking alternate terms, we get the two periods, each of eight terms,

$$y_1=R+R^9+R^{13}+R^{15}+R^{16}+R^8+R^4+R^2,$$

$$y_2=R^3+R^{10}+R^5+R^{11}+R^{14}+R^7+R^{12}+R^6.$$

Hence $y_1+y_2=-1$. We find that $y_1y_2=4(R+\cdots+R^{16})=-4$. Thus

$$y_1,\quad y_2 \quad \text{satisfy} \quad y^2+y-4=0. \tag{24}$$

Taking alternate terms in y_1, we obtain the two periods

$$z_1 = R + R^{13} + R^{16} + R^4, \qquad z_2 = R^9 + R^{15} + R^8 + R^2.$$

Taking alternate terms in y_2, we get the two periods

$$w_1 = R^3 + R^5 + R^{14} + R^{12}, \qquad w_2 = R^{10} + R^{11} + R^7 + R^6.$$

Thus $z_1 + z_2 = y_1$, $w_1 + w_2 = y_2$. We find that $z_1 z_2 = w_1 w_2 = -1$. Hence

(25) $$z_1, \quad z_2 \quad \text{satisfy} \quad z^2 - y_1 z - 1 = 0,$$

(26) $$w_1, \quad w_2 \quad \text{satisfy} \quad w^2 - y_2 w - 1 = 0.$$

Taking alternate terms in z_1, we obtain the periods

$$v_1 = R + R^{16}, \qquad v_2 = R^{13} + R^4.$$

Now, $v_1 + v_2 = z_1$, $v_1 v_2 = w_1$. Hence

(27) $$v_1, \quad v_2 \quad \text{satisfy} \quad v^2 - z_1 v + w_1 = 0.$$

As in (8), $v_1 = 2\cos A$, $v_2 = 2\cos 4A$. Hence $v_1 > v_2 > 0$, so that $z_1 = v_1 + v_2 > 0$. Similarly,

$$w_1 = 2\cos 3A + 2\cos 5A = 2\cos\frac{6\pi}{17} - 2\cos\frac{7\pi}{17} > 0,$$

$$y_2 = 2\cos 3A + 2\cos 5A + 2\cos 6A + 2\cos 7A < 0,$$

since only the first cosine in y_2 is positive and it is numerically less than the third cosine. But $y_1 y_2 = -4$, so that $y_1 > 0$. These facts prove that each of y_1, z_1, and w_1 is the single positive root of its equation (24), (25), and (26), respectively, while v_1 is the larger of the two positive roots of (27). Hence there is no ambiguity in deciding which root of (24) is y_1, which root of (25) is z_1, etc.

In § 7, we saw how to construct with ruler and compasses the roots of these quadratic equations (24)–(27) taken in this order. Since we can therefore construct a line whose length is $v_1 = 2\cos A$, we know how (§24) to construct angle $A = 2\pi/17$. But this is the angle at the center subtended by a side of a regular polygon of seventeen sides.

THEOREM 5. *We can construct with ruler and compasses a regular polygon of seventeen sides.*

Various methods have been found to obtain a single figure whose construction is equivalent to our five separate constructions of the roots of equations (24)–(27) and of angle A. Any such single figure* is necessarily very complicated and is not needed for the proof of Theorem 5.

108. General Theory† of Regular Polygons. First, let n be a prime number >2 such that $n-1$ is a power 2^h of 2 (which is true when $n=3$, 5 or 17). The $n-1$ imaginary nth roots of unity can be separated into two sets each of 2^{h-1} roots, and each such set can be subdivided into two new sets each of 2^{h-2} roots, etc., until we reach the sets R and $1/R$, R^2 and $1/R^2$, etc. This separation into sets can be done in such a manner that the periods (each being the sum of the roots in a set) satisfy quadratic equations, which are said to form a series of equations when taken in the order of our formation of the sets. The coefficients of the first equation are integers, those of the second equation involve the roots of the first equation, and in general the coefficients of any such equation involve only the roots of the equations which precede it in the series. It can be shown that each such equation has real roots. The final equation yields $R+1/R=2\cos(2\pi/n)$. Since we can construct with ruler and compasses the roots of each equation, as well as angle $2\pi/n$, we can so construct a regular polygon of n sides, provided n is a prime of the form 2^h+1.

The last property requires that h be a power 2^t of 2 (see Problem 1 below). Then for $t=0, 1, 2, 3, 4$, the numbers

$$(28) \qquad 2^{2^t}+1$$

are 3, 5, 17, 257, 65537, each being a prime number. But when $t=5$, 6, 7, 8, 9, 11, 12, 18, 23, 36, 38, or 73, the number (28) is composite.‡ For example, Euler proved in 1732 that $2^{32}+1=641\cdot 6700417$ (case $t=5$). There is no result to date for further values of t.

Second, let n be a product of distinct primes each of the form (28), or 2^k times such a product (for example, $n=15$, 30, or 60), or finally

* The simplest of such figures is given in *First Course*, page 43.

† See the author's article "Constructions with ruler and compasses; regular polygons" in *Monographs on Topics of Modern Mathematics*, Longmans, Green and Co., 1911, pp. 352–386.

‡ See the author's *History of the Theory of Numbers*, published by the Carnegie Institution of Washington, Vol. 1 (1919), pp. 375-380.

$n=2^m(m>1)$. It follows readily (see Problems 2, 3) from our first case that we can construct with ruler and compasses a regular polygon of n sides.

Third, it is impossible to so construct a regular polygon of n sides for all remaining values of n (for example, $n=7$ or 9; see Chapter IV).

PROBLEMS

1. If 2^h+1 is a prime, then h is a power of 2. Hint: Exhibit a factor when h is divisible by an odd number.

2. If two integers a and b are relatively prime, it is proved early in every book on the theory of numbers that we can find integers c and d such that $ac+bd=1$. Show that if regular polygons of a and b sides can be constructed and hence also the angles $2\pi/a$ and $2\pi/b$, then angle $2\pi/(ab)$ can be constructed and therefore also a regular polygon of ab sides.

3. Starting with a square, how do you construct in turn regular polygons of 8, 16, $\cdots$, 2^m sides?

4. List the integers <100 each of which is the number of sides of a constructable regular polygon.

5. Treat Problem 4 for the odd integers between 100 and 2000.

109. General Theory of Constructions. The first step in the consideration of a problem proposed for construction consists in formulating the problem analytically. In some instances elementary algebra suffices for this formulation. For example, in the ancient problem of the duplication of a cube, we take as a unit of length a side of the given cube, and seek the length x of a side of another cube whose volume is double that of the given cube; hence $x^3=2$.

But usually it is convenient to employ analytic geometry. This was done in § 7, where we constructed the roots of a quadratic equation with known real coefficients, provided its roots are real. A point is determined by its coordinates x and y with reference to fixed rectangular axes. A straight line is determined by an equation of the first degree, and a circle by a certain equation of the second degree. Hence we are concerned with certain numbers, some being the coordinates of points, others being the coefficients of equations, and still others expressing lengths (in terms of a given unit of length), areas, or volumes. These numbers may be said to define analytically the various geometric elements involved.

THEOREM 6. *A proposed construction is possible by ruler and compasses if and only if the numbers which define analytically the desired geometric*

elements can be derived from those defining the given elements by a finite number of rational operations and extractions of real square roots.

Proof. It is to be understood that these operations are performed at the outset upon numbers defining given elements, and second upon numbers obtained by these initial operations, and third upon numbers resulting in the second step, etc.

In § 25 we have already proved part of Theorem 6 (the "only if" part). It remains to prove the other ("if") part.

Hence we grant the condition stated in the theorem and shall prove that the construction is possible with ruler and compasses. A rational function of given quantities is obtained from them by additions, subtractions, multiplications, and divisions. The construction (by juxtaposition) of the sum or difference of two segments is obvious. When a unit of length is given, the construction, by means of parallel lines, of a segment whose length p is equal to the product $a \cdot b$ of the lengths of two given segments is shown in Fig. 28; that for the quotient $q=a/b$ in Fig. 7 of § 30. Finally, a segment of length $\sqrt{p}$ was constructed in Problem 6 of § 7. Hence the proposed construction is possible by ruler and compasses.

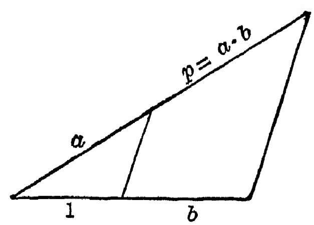


FIG. 28

EXAMPLE 1. It is impossible to construct with ruler and compasses lines representing the edges of a rectangular parallelopiped having a diagonal of length 5, surface area 24, and volume 5.

Solution. Denote the lengths of the edges by a, b, c. Then

$$a^2+b^2+c^2=25, \qquad 2ab+2ac+2bc=24, \qquad abc=5.$$

By addition we obtain from the first two equations

$$(a+b+c)^2=49, \qquad a+b+c=+7.$$

Hence a, b, c are the roots of $x^3-7x^2+12x-5=0$. By § 35, its discriminant is 169. Hence there are three distinct real roots (§ 36). Any rational root must be an integer which divides 5. By trial, no one of ± 1, ± 5 are roots. Hence there is no rational root. To complete the discussion apply Theorem 1 of Chapter IV.

PROBLEMS

Prove that it is impossible, with ruler and compasses,

1. To construct a straight line representing the distance from the circular base of a hemisphere to the parallel plane which bisects the hemisphere. *Ans.* (17).

2. To construct lines representing the lengths of the edges of an existing rectangular parallelopiped having a diagonal of length 5, surface area 24, and volume 1, 2, or 3.

Prove algebraically that it is possible, with ruler and compasses,

3. To construct every real root of $x^4+ax^2+b=0$, given lines of lengths a and b.

4. To construct the legs of a right triangle, given its area Δ and hypotenuse c. *Ans.* Square of legs $=\frac{1}{2}(c^2\pm\sqrt{c^4-16\Delta^2})$.

5. To construct the third side of a triangle, given two sides a and b and its area Δ. *Ans.* $\sqrt{(a^2+b^2\pm2\sqrt{a^2b^2-4\Delta^2})}$.

6. To locate the point P on the side $BC=1$ of a given square $ABCD$ such that the straight line AP cuts DC produced at a point Q for which the length of PQ is a given number g. Show that $y=BP$ is a root of the reciprocal equation $y^4-2y^3+(2-g^2)y^2-2y+1=0$. Find its positive roots if $g=10$.

APPENDIX

THE FUNDAMENTAL THEOREM ON SYMMETRIC FUNCTIONS AND THE FUNDAMENTAL THEOREM OF ALGEBRA

THEOREM. *Any polynomial* S *which is symmetric in* $x_1, \cdots, x_n$ *is equal to a polynomial, with integral coefficients, in the coefficients of* S *and the elementary symmetric functions*

(1) $$E_1 = \Sigma x_1, \quad E_2 = \Sigma x_1 x_2, \quad E_3 = \Sigma x_1 x_2 x_3, \cdots, \quad E_n = x_1 x_2 \cdots x_n.$$

Proof. A polynomial is called homogeneous if it is a sum of terms

$$h = a x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

each having the same total degree $k = k_1 + k_2 + \cdots + k_n$ in the x's. If a polynomial is not homogeneous it is evidently a sum of homogeneous polynomials. Hence it suffices to prove the theorem for every homogeneous symmetric polynomial S.

We may assume that no two terms of S have the same set of exponents $k_1, \cdots, k_n$ (since such terms may be combined into a single one). We shall say that h is *higher* than the term $b x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ if $k_1 > l_1$, or if $k_1 = l_1$, $k_2 > l_2$, or if $k_1 = l_1$, $k_2 = l_2$, $k_3 > l_3, \cdots$, so that the first one of the differences $k_1 - l_1$, $k_2 - l_2$, $k_3 - l_3, \cdots$ which is not zero is positive.

We first prove that, if the above term h is the highest term of S, then

$$k_1 \geqq k_2 \geqq k_3 \cdots \geqq k_n.$$

For, if $k_1 < k_2$, the symmetric polynomial S would contain the term

$$a x_1^{k_2} x_2^{k_1} x_3^{k_3} \cdots x_n^{k_n},$$

which is higher than h. If $k_2 < k_3$, S would contain the term

$$a x_1^{k_1} x_2^{k_3} x_3^{k_2} \cdots x_n^{k_n},$$

which is higher than h, etc.

If the highest term in another homogeneous symmetric polynomial S' is

$$h'=a'x_1^{k_1'}x_2^{k_2'}\cdots x_n^{k_n'},$$

and that of S is h, then the highest term in their product SS' is

$$hh'=aa'x_1^{k_1+k_1'}\cdots x_n^{k_n+k_n'}.$$

To prove this, suppose that SS' has a term, higher than hh':

$$cx_1^{l_1+l_1'}\cdots x_n^{l_n+l_n'}, \tag{2}$$

which either is a product of terms

$$t=bx_1^{l_1}\cdots x_n^{l_n}, \quad t'=b'x_1^{l_1'}\cdots x_n^{l_n'}$$

of S and S' respectively, or is a sum of such products. Since (2) is higher than hh', the first one of the differences

$$l_1+l_1'-k_1-k_1', \cdots, l_n+l_n'-k_n-k_n'$$

which is not zero is positive. But, either all the differences l_1-k_1, $\cdots$, l_n-k_n are zero or the first one which is not zero is negative, since h is either identical with t or is higher than t. Likewise for the differences $l_1'-k_1'$, $\cdots$, $l_n'-k_n'$. We therefore have a contradiction.

It follows at once that the highest term in a product of any number of homogeneous symmetric polynomials is the product of their highest terms. Now the highest terms in E_1, E_2, E_3, $\cdots$, E_n, given by (1) are,

$$x_1, \quad x_1x_2, \quad x_1x_2x_3, \quad \cdots, \quad x_1x_2\cdots x_n,$$

respectively. Hence the highest term in $E_1^{a_1}E_2^{a_2}\cdots E_n^{a_n}$ is

$$x_1^{a_1+a_2+\cdots+a_n}x_2^{a_2+\cdots+a_n}\cdots x_n^{a_n}.$$

Thus the highest term in

$$\sigma=aE_1^{k_1-k_2}E_2^{k_2-k_3}\cdots E_{n-1}^{k_{n-1}-k_n}E_n^{k_n}$$

is h. Hence $S_1=S-\sigma$ is a homogeneous symmetric polynomial of the same total degree k as S and having a highest term h_1 not as high as h. As before, we form a product σ_1 of the E's whose highest term is this h_1. Then $S_2=S_1-\sigma_1$ is a homogeneous symmetric polynomial of total degree k and with a highest term h_2 not as high as h_1. We must finally reach

a difference $S_t-\sigma_t$ which is identically zero. Indeed, there is only a finite number of products of powers of $x_1, \cdots, x_n$ of total degree k. Among these are the parts $h', h'_1, h'_2, \cdots$, of $h, h_1, h_2, \cdots$ with the coefficients suppressed. Since each h_i is not as high as h_{i-1}, the $h', h'_1, h'_2, \cdots$ are all distinct. Hence there is only a finite number of h_i. Since $S_t-\sigma_t\equiv 0$,

$$S=\sigma+S_1=\sigma+\sigma_1+S_2=\cdots=\sigma+\sigma_1+\sigma_2+\cdots+\sigma_t.$$

Hence S is a polynomial in $E_1, E_2, \cdots, E_r$ and $a, b, \cdots$, with integral coefficients.

FUNDAMENTAL THEOREM OF ALGEBRA. *Every equation*

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_n=0$$

has a complex (real or imaginary) root.

Write $z=x+iy$ where x and y are real, and similarly $a_1=c_1+id_1$, etc. By means of the binomial theorem, we may express any power of z in the form $X+iY$. Hence

$$f(z)=\phi(x,y)+i\psi(x,y), \tag{3}$$

where ϕ and ψ are polynomials with real coefficients.

LEMMA 1. $a_1h+a_2h^2+\cdots+a_nh^n$ *is less in absolute value than any assigned positive number* p *for all complex values of* h *sufficiently small in absolute value.*

The proof differs from that of Theorem 2 of § 46 only in reading "in absolute value" for "numerically" or "in numerical value."

We shall write $|z|$ for the absolute value $+\sqrt{x^2+y^2}$ of $z=x+iy$.

LEMMA 2. *Given any positive number* P, *we can find a positive number* R *such that* $|f(z)|>P$ *if* $|z|\geqq R$.

The proof is analogous to that in § 48. We have

$$f(z)=z^n(1+D), \qquad D=a_1\left(\frac{1}{z}\right)+\cdots+a_n\left(\frac{1}{z}\right)^n.$$

Since the absolute value of a sum of two complex numbers is equal to or greater than the difference of their absolute values, we have

$$|f(z)|\geqq|z|^n[1-|D|].$$

Let p be any assigned positive number <1. Applying Lemma 1 with h replaced by $1/z$, we see that $|D|<p$ if $|1/z|$ is sufficiently small, i.e., if $\rho=|z|$ is sufficiently large. Then

$$|f(z)|>\rho^n(1-p)\geqq P$$

if $\rho^n\geqq P/(1-p)$, which is true if $\rho\geqq R$, where R is the positive real nth root of $P/(1-p)$. This proves Lemma 2.

LEMMA 3. *Given a complex number* a *such that* $\mathrm{f(a)}\neq 0$, *we can find a complex number* z *for which* $|\mathrm{f(z)}|<|\mathrm{f(a)}|$.

Proof. Write $z=a+h$. By Taylor's formula (7) of, § 45,

$$f(a+h)=f(a)+f'(a)h+\cdots+f^{(r)}(a)\cdot\frac{h^r}{r!}+\cdots+f^{(n)}(a)\cdot\frac{h^n}{n!}.$$

Not all the values $f'(a)$, $f''(a)$, $\cdots$ are zero since $f^{(n)}(a)=n!$. Let $f^{(r)}(a)$ be the first one of these values which is not zero. Then

$$\frac{f(a+h)}{f(a)}=1+\frac{f^{(r)}(a)}{f(a)}\cdot\frac{h^r}{r!}+\cdots+\frac{f^{(n)}(a)}{f(a)}\cdot\frac{h^n}{n!}.$$

Writing the second member in the simpler notation

$$g(h)=1+bh^r+ch^{r+1}+\cdots+lh^n, \qquad b\neq 0,$$

we shall prove that a complex value of h may be found such that $|g(h)|<1$. Then the absolute value of $f(z)/f(a)$ will be <1 and Lemma 3 will be proved. To find such a value of h, write h and b in their trigonometric forms (§ 3)

$$h=\rho(\cos\theta+i\sin\theta), \qquad b=|b|(\cos\beta+i\sin\beta).$$

Then by formulas (3) of § 4 and (2) of § 3,

$$bh^r=|b|\rho^r\{\cos(\beta+r\theta)+i\sin(\beta+r\theta)\}.$$

Since h is at our choice, ρ and angle θ are at our choice. We choose θ so that $\beta+r\theta=180°$. Then the quantity in brackets reduces to -1, whence

$$g(h)=1-|b|\rho^r+h^r(ch+\cdot$$

By Lemma 1, we may choose ρ so small that

By taking ρ still smaller if necessary, we may assume at the same time that $|b|\rho^r<1$. Then

$$|g(h)|<(1-|b|\rho^r)+\rho^r|b|, \qquad |g(h)|<1.$$

Minimum Value of a Continuous Function. Let $F(x)$ be any polynomial with real coefficients. Among the real values of x for which $2 \leqq x \leqq 3$, there is at least one value x_1 for which $F(x)$ takes its minimum value $F(x_1)$, i.e., for which $F(x_1) \leqq F(x)$ for all real values of x such that $2 \leqq x \leqq 3$. This becomes intuitive geometrically. The portion of the graph of $y=F(x)$ which extends from its point with the abscissa 2 to its point with the abscissa 3 either has a lowest point or else has several equally low points, each lower than all the remaining points. The arithmetic proof depends upon the fact that $F(x)$ is continuous for each x between 2 and 3 inclusive (§ 46). The proof is rather delicate and is omitted since the theorem for functions of one variable x is mentioned here only by way of introduction to our case of functions of two variables.

We are interested in the analogous question for

$$G(x, y)=\phi^2(x, y)+\psi^2(x, y),$$

which, by (3), is the square of $|f(z)|$. As in the elements of solid analytic geometry, consider the surface represented by $z=G(x, y)$ and the right circular cylinder $x^2+y^2=R^2$. Of the points on the first surface and on or within their curve of intersection there is a lowest point or there are several equally low lowest points, possibly an infinite number of them. Expressed arithmetically, among all the pairs of real numbers x, y for which $x^2+y^2 \leqq R^2$, there is* at least one pair x_1, y_1 for which the polynomial $G(x, y)$ takes a minimum value $G(x_1, y_1)$, i.e., for which $G(x_1, y_1) \leqq G(x, y)$ for all pairs of real numbers x, y for which $x^2+y^2 \leqq R^2$.

Proof of the Fundamental Theorem. Let z' denote any complex number for which $f(z') \neq 0$. Let P denote any positive number exceeding $|f(z')|$. Determine R as in Lemma 2. In it the condition $|z| \geqq R$ may be interpreted geometrically to imply that the point (x, y) representing $z=x+iy$ is outside or on the circle C having the equation $x^2+y^2=R^2$.

* Harkness and Morley, *Introduction to the Theory of Analytic Functions*, p. 79, prove that a real function of two variables which is continuous throughout a closed region has a minimum value at some point of the region.

Lemma 2 thus states that, if z is represented by any point outside or on the circle C, then $|f(z)|>P$. In other words, if $|f(z)|\leqq P$, the point representing z is inside circle C. In particular, the point representing z' is inside circle C.

In view of the preceding section on minimum value, we have

for all pairs of real numbers x, y for which $x^2+y^2\leqq R^2$, where x_1, y_1 is one such pair. Write z_1 for x_1+iy_1. Since $|f(z)|^2=G(x, y)$, we have

for all z's represented by points on or within circle C. Since z' is represented by such a point,

(4) $$|f(z_1)|\leqq|f(z')|<P.$$

This number z_1 is a root of $f(z)=0$. For, if $f(z_1)\neq 0$, Lemma 3 shows that there would exist a complex number z for which

(5) $$|f(z)|<|f(z_1)|.$$

Then $|f(z)|<P$ by (4), so that the point representing z is inside circle C, as shown above. By the statement preceding (4),

But this contradicts (5). Hence the fundamental theorem is proved.

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